the total amount spent in the store by all customers arriving by time $t$, then 
$\{X(t), t \geq 0\}$ is a compound Poisson process.

2.6 Conditional Poisson Processes

Let $\Lambda$ be a positive random variable having distribution $G$ and let $\{N(t), t \geq 0\}$ be a counting process such that, given that $\Lambda = \lambda$, $\{N(t), t \geq 0\}$ is a Poisson process having rate $\lambda$. Thus, for instance,

$$P[N(t + s) - N(s) = n] = \int_0^\infty e^{-\lambda} \frac{(\lambda t)^n}{n!} dG(\lambda).$$

The process $\{N(t), t \geq 0\}$ is called a conditional Poisson process since, conditional on the event that $\Lambda = \lambda$, it is a Poisson process with rate $\lambda$. It should be noted, however, that $\{N(t), t \geq 0\}$ is not a Poisson process. For instance, whereas it does have stationary increments, it does not have independent ones. (Why not?)

Let us compute the conditional distribution of $\Lambda$ given that $N(t) = n$. For $d\lambda$ small,

$$P[\Lambda \in (\lambda, \lambda + d\lambda) | N(t) = n] = \frac{P[N(t) = n | \Lambda \in (\lambda, \lambda + d\lambda)] P[\Lambda \in (\lambda, \lambda + d\lambda)]}{P[N(t) = n]}$$

$$= \frac{e^{-\lambda} \frac{(\lambda t)^n}{n!} dG(\lambda)}{\int_0^\infty e^{-\lambda} \frac{(\lambda t)^n}{n!} dG(\lambda)}$$

and so the conditional distribution of $\Lambda$, given that $N(t) = n$, is given by

$$P[\Lambda \leq x | N(t) = n] = \frac{\int_0^x e^{-\lambda} \frac{(\lambda t)^n}{n!} dG(\lambda)}{\int_0^\infty e^{-\lambda} \frac{(\lambda t)^n}{n!} dG(\lambda)}.$$

Example 2.6(a) Suppose that, depending on factors not at present understood, the average rate at which seismic shocks occur in a certain region over a given season is either $\lambda_1$ or $\lambda_2$. Suppose also that it is $\lambda_1$ for $100 p$ percent of the seasons and $\lambda_2$ the remaining time. A simple model for such a situation would be to suppose that $\{N(t), 0 \leq t < \infty\}$ is a conditional Poisson process such that $\Lambda$ is either $\lambda_1$ or $\lambda_2$ with respective probabilities $p$ and $1 - p$. Given $n$ shocks in the first $t$ time units of a season, then the probability it is a $\lambda_1$ season is

$$P[\Lambda = \lambda_1 | N(t) = n] = \frac{pe^{-\lambda_1}(\lambda_1 t)^n}{pe^{-\lambda_1}(\lambda_1 t)^n + e^{-\lambda_2}(\lambda_2 t)^n(1 - p)}.$$

Also, by conditioning on whether $\Lambda = \lambda_1$ or $\Lambda = \lambda_2$, we see that the time from $t$ until the next shock, given $N(t) = n$, has the distribution

$$P[\text{time from } t \text{ until next shock is } \leq x | N(t) = n] = \frac{p(1 - e^{-\lambda_1 t})e^{-\lambda_1(\lambda_1 t)^n} + (1 - e^{-\lambda_2 t})e^{-\lambda_2(\lambda_2 t)^n}(1 - p)}{pe^{-\lambda_1(\lambda_1 t)^n} + e^{-\lambda_2(\lambda_2 t)^n}(1 - p)}.$$

Problems

2.1. Show that Definition 2.1.1 of a Poisson process implies Definition 2.1.2.

2.2. For another approach to proving that Definition 2.1.2 implies Definition 2.1.1:

(a) Prove, using Definition 2.1.2, that

$$P_N(t + s) = P_N(t)P_N(s).$$

(b) Use (a) to infer that the interarrival times $X_1, X_2, \ldots$ are independent exponential random variables with rate $\lambda$.

(c) Use (b) to show that $N(t)$ is Poisson distributed with mean $\lambda t$.

2.3. For a Poisson process show, for $s < t$, that

$$P[N(s) = k | N(t) = n] = \binom{n}{k} \left( \frac{1}{t} \right)^k \left( 1 - \frac{1}{t} \right)^{n-k}, \quad k = 0, 1, \ldots, n.$$  

2.4. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda$. Calculate $E[N(t) \cdot N(t + s)]$.

2.5. Suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates $\lambda_1$ and $\lambda_2$. Show that $\{N(t) + N_1(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$. Also, show that the probability that the first event of the combined process comes from $\{N_1(t), t \geq 0\}$ is $\lambda_1 / (\lambda_1 + \lambda_2)$, independently of the time of the event.

2.6. A machine needs two types of components in order to function. We have a stockpile of $n$ type-1 components and $m$ type-2 components.
2.7. Compute the joint distribution of $S_1, S_2, S_3$.

2.8. Generating a Poisson Random Variable. Let $U_1, U_2, \ldots$ be independent uniform $(0, 1)$ random variables.

(a) If $X_t = (\log U_t)/\lambda$, show that $X_t$ is exponentially distributed with rate $\lambda$.

(b) Use part (a) to show that $N$ is Poisson distributed with mean $\lambda$ when $N$ is defined to equal that value of $n$ such that

$$\prod_{i=1}^{n} U_i = e^{-\lambda} > \prod_{i=1}^{n+1} U_i,$$

where $\prod_{i=1}^{n} U_i = 1$. Compare with Problem 1.21 of Chapter 1.

2.9. Suppose that events occur according to a Poisson process with rate $\lambda$. Each time an event occurs we must decide whether or not to stop, with our objective being to stop at the last event to occur prior to some specified time $T$. That is, if an event occurs at time $t$, $0 \leq t \leq T$ and we decide to stop, then we lose if there are no events in the interval $(t, T]$, and win otherwise. If we do not stop when an event occurs, and no additional events occur by time $T$, then we also lose. Consider the strategy that stops at the first event that occurs after some specified time $s$, $0 \leq s \leq T$.

(a) If the preceding strategy is employed, what is the probability of winning?

(b) What value of $s$ maximizes the probability of winning?

(c) Show that the probability of winning under the optimal strategy is $1/e$.

2.10. Buses arrive at a certain stop according to a Poisson process with rate $\lambda$. If you take the bus from that stop then it takes a time $R$, measured from the time at which you enter the bus, to arrive home. If you walk from the bus stop then it takes a time $W$ to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to a time $s$, and if a bus has not yet arrived by that time then you walk home.

(a) Compute the expected time from when you arrive at the bus stop until you reach home.

(b) Show that if $W < 1/\lambda + R$ then the expected time of part (a) is minimized by letting $s = 0$; if $W > 1/\lambda + R$ then it is minimized by letting $s = \infty$ (that is, you continue to wait for the bus); and when $W = 1/\lambda + R$ all values of $s$ give the same expected time.

(c) Give an intuitive explanation of why we need only consider the cases $s = 0$ and $s = \infty$ when minimizing the expected time.

2.11. Cars pass a certain street location according to a Poisson process with rate $\lambda$. A person wanting to cross the street at that location waits until she can see that no cars will come by in the next $T$ time units. Find the expected time that the person waits before starting to cross. (Note, for instance, that if no cars will be passing in the first $T$ time units then the waiting time is 0.)

2.12. Events, occurring according to a Poisson process with rate $\lambda$, are registered by a counter. However, each time an event is registered the counter becomes inoperative for the next $b$ units of time and does not register any new events that might occur during that interval. Let $R(t)$ denote the number of events that occur by time $t$ and are registered.

(a) Find the probability that the first $k$ events are all registered.

(b) For $t \geq (n-1)b$, find $P[R(t) \geq n]$.

2.13. Suppose that shocks occur according to a Poisson process with rate $\lambda$, and suppose that each shock, independently, causes the system to fail with probability $p$. Let $N$ denote the number of shocks that it takes for the system to fail and let $T$ denote the time of failure. Find $P[N = n \mid T = t]$.

2.14. Consider an elevator that starts in the basement and travels upward. Let $N_i$ denote the number of people that get in the elevator at floor $i$. Assume the $N_i$ are independent and that $N_i$ is Poisson with mean $\lambda_i$. Each person entering at $i$ will, independent of everything else, get off at $j$ with probability $P_{ij}, \sum_j P_{ij} = 1$. Let $O_i$ = number of people getting off the elevator at floor $j$.

(a) Compute $E[O_j]$.

(b) What is the distribution of $O_j$?

(c) What is the joint distribution of $O_j$ and $O_k$?

2.15. Consider an $r$-sided coin and suppose that on each flip of the coin exactly one of the sides appears: side $i$ with probability $P_i, \sum_i P_i = 1$. For given numbers $n_1, \ldots, n_r$, let $N_i$ denote the number of flips required until side $i$ has appeared for the $n_i$ time, $i = 1, \ldots, r$, and let

$$N = \min_{i=1}^{r} N_i.$$
Thus $N$ is the number of flips required until side $i$ has appeared $n_i$ times for some $i = 1, \ldots, r$.

(a) What is the distribution of $N$?

(b) Are the $N_i$ independent?

Now suppose that the flips are performed at random times generated by a Poisson process with rate $\lambda = 1$. Let $T_i$ denote the time until side $i$ has appeared for the $n_i$ time, $i = 1, \ldots, r$ and let

$$T = \min_{i=1,\ldots,r} T_i.$$

(c) What is the distribution of $T$?

(d) Are the $T_i$ independent?

(e) Derive an expression for $E[T]$.

(f) Use (e) to derive an expression for $E[N]$.

2.16. The number of trials to be performed is a Poisson random variable with mean $\lambda$. Each trial has $n$ possible outcomes and, independent of everything else, results in outcome number $i$ with probability $p_i = \frac{1}{n-1}$, $i = 0, 1, \ldots, n$. Let $X_i$ denote the number of outcomes that occur exactly $j$ times, $j = 0, 1, \ldots$. Compute $E[X_i]$, Var($X_i$).

2.17. Let $X_1, X_2, \ldots, X_n$ be independent continuous random variables with common density function $f$. Let $X_{(i)}$ denote the $i$th smallest of $X_1, \ldots, X_n$.

(a) Note that in order for $X_{(i)}$ to equal $x$, exactly $i-1$ of the $X_i$'s must be less than $x$, one must equal $x$, and the other $n-i$ must all be greater than $x$. Using this fact argue that the density function of $X_{(i)}$ is given by

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!n!} (F(x))^{i-1}(1-F(x))^{n-i}f(x).$$

(b) $X_{(i)}$ will be less than $x$ if, and only if, how many of the $X_i$'s are less than $x$?

(c) Use (b) to obtain an expression for $P[X_{(i)} \leq x]$.

(d) Using (a) and (c) establish the identity

$$\sum_{k=i}^{n} \binom{n}{k} y^k(1-y)^{n-k} = \int_0^x \frac{n!}{(i-1)!n!} x^{i-1}(1-x)^{n-i} dx$$

for $0 \leq y \leq 1$.

2.18. Let $U_1, \ldots, U_n$ denote the order statistics of a set of $n$ uniform $(0, 1)$ random variables. Show that given $U_{(i)} = y$, $U_{(i+1)}, \ldots, U_{(n)}$ are distributed as the order statistics of a set of $n-1$ uniform $(0, y)$ random variables.

2.19. Busloads of customers arrive at an infinite server queue at a Poisson rate $\lambda$. Let $G$ denote the service distribution. A bus contains $j$ customers with probability $\alpha_j$, $j = 1, \ldots$. Let $X(t)$ denote the number of customers that have been served by time $t$.

(a) $E[X(t)] = ?$

(b) Is $X(t)$ Poisson distributed?

2.20. Suppose that each event of a Poisson process with rate $\lambda$ is classified as being either of type $1, 2, \ldots, k$. If the event occurs at $s$, then, independently of all else, it is classified as type $i$ with probability $P_i(s) = \frac{1}{k}$, $i = 1, \ldots, k$. Let $N_i(t)$ denote the number of type $i$ arrivals in $[0, t]$. Show that the $N_i(t), i = 1, \ldots, k$ are independent and $N(t)$ is Poisson distributed with mean $\lambda \int_0^t \sum_{i=1}^k P_i(s) ds$.

2.21. Individuals enter the system in accordance with a Poisson process having rate $\lambda$. Each arrival independently makes its way through the states of the system. Let $\alpha_i(s)$ denote the probability that an individual in state $i$ a time $s$ after it arrived. Let $N_i(t)$ denote the number of individuals in state $i$ at time $t$. Show that the $N_i(t), i \geq 1$, are independent and $N(t)$ is Poisson with mean equal to

$$\lambda E[\text{amount of time an individual is in state } i \text{ during its first } t \text{ units in the system}].$$

2.22. Suppose cars enter a one-way infinite highway at a Poisson rate $\lambda$. The $i$th car to enter chooses a velocity $V_i$ and travels at this velocity. Assume that the $V_i$'s are independent positive random variables having a common distribution $F$. Derive the distribution of the number of cars that are located in the interval $(a, b)$ at time $t$. Assume that no time is lost when one car overtakes another car.

2.23. For the model of Example 2.3(C), find

(a) $\text{Var}[D(t)]$.

(b) $\text{Cov}[D(t), D(t+s)]$. 

2.24. Suppose that cars enter a one-way highway of length \( L \) in accordance with a Poisson process with rate \( \lambda \). Each car travels at a constant speed that is randomly determined, independently from car to car, from the distribution \( F \). When a faster car encounters a slower one, it passes it with no loss of time. Suppose that a car enters the highway at time \( t \). Show that as \( t \to \infty \) the speed of the car that minimizes the expected number of encounters with other cars, where we say an encounter occurs when a car is either passed by or passes another car, is the median of the distribution \( G \).

2.25. Suppose that events occur in accordance with a Poisson process with rate \( \lambda \), and that an event occurring at time \( s \) independent of the past, contributes a random amount having distribution \( F \), \( s \geq 0 \). Show that \( W \), the sum of all contributions by time \( t \), is a compound Poisson random variable. That is, show that \( W \) has the same distribution as \( \sum_{i=1}^{N} X_i \), where the \( X_i \) are independent and identically distributed random variables and are independent of \( N \), a Poisson random variable. Identify the distribution of the \( X_i \) and the mean of \( N \).

2.26. Compute the conditional distribution of \( S_1, S_2, \ldots, S_n \) given that \( S_n = t \).

2.27. Compute the moment generating function of \( D(t) \) in Example 2.3(C).

2.28. Prove Lemma 2.3.3.

2.29. Complete the proof that for a nonhomogeneous Poisson process \( N(t+s) - N(t) \) is Poisson with mean \( m(t+s) - m(t) \).

2.30. Let \( T_1, T_2, \ldots \) denote the interarrival times of events of a nonhomogeneous Poisson process having intensity function \( \lambda(t) \).
   (a) Are the \( T_i \) independent?
   (b) Are the \( T_i \) identically distributed?
   (c) Find the distribution of \( T_1 \).
   (d) Find the distribution of \( T_2 \).

2.31. Consider a nonhomogeneous Poisson process \( \{N(t), t \geq 0\} \), where \( \lambda(t) > 0 \) for all \( t \). Let
   \[ N^*(t) = N(m^{-1}(t)) \]
   Show that \( \{N^*(t), t \geq 0\} \) is a Poisson process with rate \( \lambda = 1 \).

2.32. (a) Let \( \{N(t), t \geq 0\} \) be a nonhomogeneous Poisson process with mean value function \( m(t) \). Given \( N(t) = n \), show that the unordered set of arrival times has the same distribution as \( n \) independent and identically distributed random variables having distribution function
   \[
   F(x) = \begin{cases} 
   m(x) & x \leq t \\
   m(t) & x > t.
   \end{cases}
   \]
   (b) Suppose that workers incur accidents in accordance with a nonhomogeneous Poisson process with mean value function \( m(t) \). Suppose further that each injured person is out of work for a random amount of time having distribution \( F \). Let \( X(t) \) be the number of workers who are out of work at time \( t \). Compute \( E[X(t)] \) and \( Var(X(t)) \).

2.33. A two-dimensional Poisson process is a process of events in the plane such that (i) for any region of area \( A \), the number of events in \( A \) is Poisson distributed with mean \( AA \), and (ii) the numbers of events in nonoverlapping regions are independent. Consider a fixed point, and let \( X \) denote the distance from that point to its nearest event, where distance is measured in the usual Euclidean manner. Show that:
   (a) \( P[X > r] = e^{-\pi r^2} \).
   (b) \( E[X] = \frac{1}{2\sqrt{\lambda}} \).
   Let \( R_i, i \geq 1 \) denote the distance from an arbitrary point to the \( i \)th closest event to it. Show that, with \( R_0 = 0 \),
   (c) \( \pi R_i^2 - \pi R_{i-1}^2, i \geq 1 \) are independent exponential random variables, each with rate \( \lambda \).

2.34. Repeat Problem 2.25 when the events occur according to a nonhomogeneous Poisson process with intensity function \( \lambda(t), t \geq 0 \).

2.35. Let \( \{N(t), t \geq 0\} \) be a nonhomogeneous Poisson process with intensity function \( \lambda(t), t \geq 0 \). However, suppose one starts observing the process at a random time \( \tau \) having distribution function \( F \). Let \( N^*(t) = N(\tau + t) - N(\tau) \) denote the number of events that occur in the first \( t \) time units of observation.
   (a) Does the process \( \{N^*(t), t \geq 0\} \) possess independent increments?
   (b) Repeat (a) when \( \{N(t), t \geq 0\} \) is a Poisson process.

2.36. Let \( C \) denote the number of customers served in an \( M/G/1 \) busy period. Find
   (a) \( E[C] \).
   (b) \( Var(C) \).

PROBLEMS