AMS 263 — Stochastic Processes (Fall 2005)

Homework 2 (due Thursday October 27)

For all the problems, recall the definition of a Markov chain: a discrete-time \((T = \{0, 1, 2, \ldots\})\),
discrete-state \((S \subseteq Z)\) stochastic process \(X = \{X_n : n \geq 0\}\) is a Markov chain if and only if it
satisfies the Markov condition

\[ P(X_{n+1} = s \mid X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = P(X_{n+1} = s \mid X_n = x_n) \]  

(1)

for all \(n \geq 1\) and all \(s, x_0, x_1, \ldots, x_n \in S\). Also, note that we assume that \(X\) is a time-homogeneous
Markov chain unless explicitly stated otherwise.

1. Show that the Markov property (1) is equivalent to each of the following conditions:
   (a) For all \(n, m \geq 1\) and all \(s, x_0, x_1, \ldots, x_n \in S\),
   \[ P(X_{n+m} = s \mid X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = P(X_{n+m} = s \mid X_n = x_n). \]
   (b) For all \(0 \leq n_1 < n_2 < \ldots < n_k \leq n\), all \(m \geq 1\) and all \(s, x_1, \ldots, x_k \in S\),
   \[ P(X_{n+m} = s \mid X_{n_1} = x_1, X_{n_2} = x_2, \ldots, X_{n_k} = x_k) = P(X_{n+m} = s \mid X_{n_k} = x_k). \]
   (c) For all \(n > 1\), \(x_n \in S\), all finite sets of time points \(\{\ell \in L : 1 \leq \ell < n\}\) and \(\{k \in K : n < k\}\),
   and all corresponding sets of states \(\{x_k : k \in K\}\), \(\{x_\ell : \ell \in L\}\),
   \[ P(\{X_k = x_k : k \in K\}, \{X_\ell = x_\ell : \ell \in L\} \mid X_n = x_n) = P(\{X_k = x_k : k \in K\} \mid X_n = x_n)P(\{X_\ell = x_\ell : \ell \in L\} \mid X_n = x_n). \]
   (That is, “given the present, future is independent of the past”.)

2. Assume \(X = \{X_n : n \geq 0\}\) is a Markov chain and let \(\{n_k : k \geq 0\}\) be an unbounded increasing
   sequence of positive integers. Define a new stochastic process \(Y = \{Y_k : k \geq 0\}\) such that \(Y_k = X_{n_k}\).
   Show that \(Y\) is a Markov chain. Is \(Y\) a time-homogeneous Markov chain without additional
   conditions?

3. Consider a Markov chain \(X = \{X_n : n \geq 0\}\) and let \(I : S^n \to \{0, 1\}\) be a function that
   assigns the value of 0 or 1 to each collection of states \((x_1, \ldots, x_n)\). Show that, for any \(m \geq 1\),
   the distribution of \(X_{n+m}\) conditional on \(\{I(X_1, \ldots, X_n) = 1\} \cap \{X_n = i\}\) is identical to the
distribution of \(X_{n+m}\) conditional on \(\{X_n = i\}\).
4. **(Strong Markov property.)** Assume \( X = \{X_n : n \geq 0\} \) is a Markov chain. Let \( W \) be a random variable (random time), taking values in \( \{0, 1, 2, \ldots\} \), with the property that the indicator function of the event \( \{W = n\} \) is a function of the variables \( X_0, X_1, \ldots, X_n \) only. (Such a random variable \( W \) is called a *stopping time*. The definition requires that we can determine (probabilities for) its values, \( W = n \), with a knowledge only of the past and the present, \( X_0, X_1, \ldots, X_n \), and with no further information about the future.) Show that

\[
P(X_{W+m} = j \mid X_k = x_k, 0 \leq k < W, X_W = i) = P(X_{W+m} = j \mid X_W = i),
\]

for all \( m \geq 0 \) and all \( x_k, i, j \in S \).

5. Consider a Markov chain with three states \( \{1, 2, 3\} \) and transition matrix

\[
\begin{pmatrix}
1 - 2p & 2p & 0 \\
p & 1 - 2p & p \\
0 & 2p & 1 - 2p
\end{pmatrix}
\]

where \( 0 \leq p < 0.5 \). Classify the states of the chain. Obtain \( p_{ii}(n) \) for \( i = 1, 2, 3 \). Calculate the mean recurrence times \( \mu_i, i = 1, 2, 3 \), of the states.

6. Show that for an irreducible persistent (discrete-time, discrete-state) Markov chain, \( f_{ji} = 1 \) for all \( i, j \in S \).

7. Recall that the transition matrix \( P \) of a (discrete-time, discrete-state) Markov chain is stochastic, that is, all its entries \( p_{ij} \) are non-negative and \( \sum_{j \in S} p_{ij} = 1 \), for all \( i \in S \). The transition matrix is *doubly stochastic* if it also satisfies \( \sum_{i \in S} p_{ij} = 1 \), for all \( j \in S \).

(a) Consider an irreducible (discrete-time, discrete-state) Markov chain \( X \) with finite state space \( S \) of dimension \( K = |S| \) \((< \infty)\), and transition matrix \( P \). Show that \( X \) has a discrete uniform stationary distribution (that is, \( \pi_i = K^{-1} \) for each \( i \in S \)) if and only if \( P \) is doubly stochastic.

(b) Show that if a (discrete-time, discrete-state) Markov chain has finite state space and doubly stochastic transition matrix, then all its states are positive persistent.