AMS 261: Probability Theory (Spring 2011)

Homework 3 (due Thursday May 12)

1. Let $F$ and $G$ be distribution functions on $\mathbb{R}$ such that $G(t) \leq F(t)$, for all $t \in \mathbb{R}$ (in which case, $G$ is said to be stochastically larger than $F$).
   • Construct two $\mathbb{R}$-valued random variables $X$ and $Y$, defined on the same probability space $(\Omega, \mathcal{F}, P)$, such that the distribution function of $X$ is $G$, the distribution function of $Y$ is $F$, and $P(X \geq Y) = 1$.

2. Consider a simple random variable $X$ defined on some probability space $(\Omega, \mathcal{F}, P)$, and let $F$ be its distribution function. Denote by $F(x^-) = \lim_{y \to x^-} F(y)$ (or equivalently, $F(x^-) = \lim_{n \to \infty} F(x_n)$ for an arbitrary increasing sequence $\{x_n : n = 1, 2, \ldots\}$ converging to $x$).
   • Show that the expectation of $X$ can be written in the form
     \[ E(X) = \sum_{x \in \mathbb{R}} x\{F(x) - F(x^-)\}. \]

3. Let $X$ be a simple random variable (taking both negative and positive values) defined on some probability space $(\Omega, \mathcal{F}, P)$.
   • Show that expectation definitions 1 (for simple random variables) and 3 (for general random variables taking values on the extended real line) are equivalent.

4. Consider an $\mathbb{R}^+$-valued random variable $X$, defined on some probability space $(\Omega, \mathcal{F}, P)$ such that $E(X) < \infty$. Let $A = \{\omega \in \Omega : X(\omega) = +\infty\}$ (recall that, based on the general definition for $\mathbb{R}^+$-valued measurable functions, we have that $A \in \mathcal{F}$).
   • Show that $X$ is almost surely finite, that is, $P(A) = 0$.

5. Consider a sequence $\{X_n : n = 1, 2, \ldots\}$ of $\mathbb{R}^+$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that the sequence is (pointwise) increasing, that is, for all $n$ and for each $\omega \in \Omega$, $X_n(\omega) \leq X_{n+1}(\omega)$. Denote by $X$ the pointwise limit of $\{X_n : n = 1, 2, \ldots\}$, that is, for each $\omega \in \Omega$, $X(\omega) = \lim_{n \to \infty} X_n(\omega)$, and assume that $E(X) < \infty$. Define the variance for $X$ by $\text{Var}(X) = E\{(X - E(X))^2\}$, and similarly, for each $n$, $\text{Var}(X_n) = E\{(X_n - E(X_n))^2\}$. (In general, the variance for a random variable $Y$ with finite expectation $E(Y)$ is given by $\text{Var}(Y) = E\{(Y - E(Y))^2\}$, whether finite or infinite.)
   • Prove that $\text{Var}(X) = \lim_{n \to \infty} \text{Var}(X_n)$.

6. Let $\{X_n : n = 1, 2, \ldots\}$, $\{Y_n : n = 1, 2, \ldots\}$, and $\{Z_n : n = 1, 2, \ldots\}$ be sequences of $\mathbb{R}$-valued random variables (all the random variables are defined on the same probability space). Assume that: (a) $E(X_n)$ and $E(Z_n)$ exist for all $n$ and are finite; (b) each of the three sequences converges almost surely (denote by $X$, $Y$, and $Z$ the respective almost sure limits); (c) $E(X)$, $E(Y)$, and $E(Z)$ exist and are finite; (d) $X_n \leq Y_n \leq Z_n$ almost surely; (e) $\lim_{n \to \infty} E(X_n) = E(X)$, and $\lim_{n \to \infty} E(Z_n) = E(Z)$.
   • Show that $\lim_{n \to \infty} E(Y_n) = E(Y)$.