AMS 261: Probability Theory (Spring 2011)

Homework 2 (due Thursday 4/28)

1. Let \( \{A_n : n = 1, 2, \ldots\} \) be a countable sequence of subsets of a sample space \( \Omega \).
   (a) Assume that \( \{A_n : n = 1, 2, \ldots\} \) is an increasing sequence, that is, \( A_n \subseteq A_{n+1} \), for all \( n \geq 1 \).
       Show that \( \lim_{n \to \infty} A_n \) exists, and \( \lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} A_n \).
   (b) Assume that \( \{A_n : n = 1, 2, \ldots\} \) is a decreasing sequence, that is, \( A_{n+1} \subseteq A_n \), for all \( n \geq 1 \).
       Show that \( \lim_{n \to \infty} A_n \) exists, and \( \lim_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} A_n \).

2. Consider a measurable space \((\Omega, \mathcal{F})\) and a set function \( P : \mathcal{F} \to [0,1] \), which satisfies \( P(\Omega) = 1 \),
   and \( P(A \cup B) = P(A) + P(B) \) for any \( A \) and \( B \) in \( \mathcal{F} \) with \( A \cap B = \emptyset \). Moreover, assume that
   \( P \) is continuous, that is, \( P(\lim_{n \to \infty} A_n) = \lim_{n \to \infty} P(A_n) \), for any sequence \( \{A_n : n = 1, 2, \ldots\} \)
   of sets in \( \mathcal{F} \) for which \( \lim_{n \to \infty} A_n \) exists. Prove that \( P \) is a probability measure on \((\Omega, \mathcal{F})\).

3. Prove that any non-decreasing function from \( \mathbb{R} \) to \( \mathbb{R} \) is measurable. (Assume the usual Borel
   \( \sigma \)-field on \( \mathbb{R} \).)

4. Let \((\Omega_j, \mathcal{F}_j)\), \( j = 1, 2, 3 \), be measurable spaces. Consider measurable functions \( X : \Omega_1 \to \Omega_2 \) and
   \( Y : \Omega_2 \to \Omega_3 \), and define the composition function \( Y \circ X : \Omega_1 \to \Omega_3 \) by \( Y \circ X(\omega_1) = Y(X(\omega_1)) \),
   for any \( \omega_1 \in \Omega_1 \). Show that \( Y \circ X \) is a measurable function.

5. Consider a sequence \( \{X_n : n = 1, 2, \ldots\} \) of \( \mathbb{R} \)-valued random variables defined on the same
   probability space \((\Omega, \mathcal{F}, P)\). Let \( C \) be the set of \( \omega \in \Omega \) such that \( \{X_n(\omega) : n = 1, 2, \ldots\} \) is a
   convergent numerical sequence. Prove that \( C \in \mathcal{F} \).

6. Let \( X \) and \( Y \) be \( \mathbb{R} \)-valued random variables defined on the same probability space \((\Omega, \mathcal{F}, P)\),
   and consider the subset of \( \Omega \) defined by \( A = \{\omega \in \Omega : X(\omega) \neq Y(\omega)\} \).
   (a) Prove that \( A \) is an event in \( \mathcal{F} \).
   (Hint: Recall the Archimedean Property of the real numbers, according to which, for any two
   real numbers \( a \) and \( b \) with \( a < b \), there exists a rational number \( q \) such that \( a < q < b \).)
   (b) Assume that \( P(A) = 0 \). Prove that \( P(X^{-1}(B)) = P(Y^{-1}(B)) \) for any Borel subset \( B \) of \( \mathbb{R} \)
   (in which case, we say that the distributions of \( X \) and \( Y \) are equal).