1. Consider a sample space Ω.
   (a) Prove that any intersection of σ-fields (of subsets of Ω) is a σ-field. That is, if \( \mathcal{F}_j, j \in J \), are σ-fields on Ω (with \( J \) an arbitrary index set, countable or uncountable), then show that \( \mathcal{F} = \bigcap_{j \in J} \mathcal{F}_j \) is a σ-field.
   (b) Show by example that a union of σ-fields may not be a σ-field.

2. Given a sample space Ω and a collection \( \mathcal{E} \) of subsets of Ω, the σ-field generated by \( \mathcal{E} \), \( \sigma(\mathcal{E}) \), is defined as the intersection of all σ-fields on Ω that contain \( \mathcal{E} \). (Note that it can be shown that \( \sigma(\mathcal{E}) \) is indeed a σ-field, using, for instance, part (a) of problem 1 above.)
   (a) Consider collections \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) of subsets of the sample space Ω. Show that if \( \mathcal{E}_1 \subseteq \mathcal{E}_2 \), then \( \sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2) \).
   (b) As in part (a), let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be collections of subsets of the sample space Ω. Prove that if \( \mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2) \) and \( \mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1) \), then \( \sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2) \).

3. Let \( \mathcal{F} \) be a collection of subsets of a sample space Ω.
   (a) Suppose that \( \Omega \in \mathcal{F} \) and that \( A, B \in \mathcal{F} \) implies \( A \cap B^c \in \mathcal{F} \). Show that \( \mathcal{F} \) is a field.
   (b) Suppose that \( \Omega \in \mathcal{F} \) and that \( \mathcal{F} \) is closed under the formation of complements and finite pairwise disjoint unions. Show that \( \mathcal{F} \) need not be a field.

4. Consider the sample space \( \Omega = (0, 1] \) and the collection \( \mathcal{B}_0 \) of all finite pairwise disjoint unions of subintervals of \( (0, 1] \). That is, any member \( B \) of \( \mathcal{B}_0 \) is of the form \( B = \bigcup_{i=1}^n (a_i, b_i] \), where for each \( i = 1, ..., n \), \( 0 \leq a_i < b_i \leq 1 \), and \( (a_i, b_i] \cap (a_j, b_j] = \emptyset \) for any \( i \neq j \).
   Show that \( \mathcal{B}_0 \) augmented by the empty set is a field, but not a σ-field.

5. Consider a countable sequence \( \{A_n : n = 1, 2, ...\} \) of events in a probability space \( (\Omega, \mathcal{F}, P) \).
   Show that
   \[
   P \left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} P(A_n).
   \]

6. Let \( \Omega = \{\omega_1, \omega_2, \ldots\} \) be a countable set, \( \{p_n : n = 1, 2, \ldots\} \) be a sequence of non-negative real numbers such that \( \sum_{n=1}^{\infty} p_n = 1 \), and \( \mathcal{F} \) be the collection of all subsets of \( \Omega \). For each \( A \in \mathcal{F} \), define the set function
   \[
   P(A) = \sum_{\{n : \omega_n \in A\}} p_n.
   \]
   Show that \( (\Omega, \mathcal{F}, P) \) is a probability space.