1. A countable sequence \( \{X_n : n = 1, 2, \ldots \} \) of \( \mathbb{R} \)-valued random variables, defined on a common probability space \((\Omega, \mathcal{F}, P)\), is said to converge completely if for any \( k = 1, 2, \ldots \), 
\[ \sum_{n=1}^{\infty} P(|X_n| > k^{-1}) < \infty. \]
- Show that if \( \{X_n : n = 1, 2, \ldots \} \) converges completely, then \( \lim_{n \to \infty} X_n = 0 \) almost surely.

2. Construct a countable sequence \( \{X_n : n = 1, 2, \ldots \} \) of \( \mathbb{R}^+ \)-valued random variables (i.e., \( X_n \geq 0 \), for all \( n \)) that satisfies \( \sum_{n=1}^{\infty} P(X_n > k^{-1}) < \infty \), for any \( k = 1, 2, \ldots \), but for which \( \lim_{n \to \infty} E(X_n) \neq 0 \).

3. For \( k = 1, 2, \ldots \), consider random variables \( X_k : (\Omega, \mathcal{F}, P) \to (\Psi_k, \mathcal{G}_k) \) and measurable functions \( \varphi_k : (\Psi_k, \mathcal{G}_k) \to (\Theta_k, \mathcal{H}_k) \). Assume that the countable sequence of random variables \( \{X_k : k = 1, 2, \ldots \} \) is independent.
- Prove that the sequence \( \{\varphi_k \circ X_k : k = 1, 2, \ldots \} \) is independent.

4. Let \( \{A_n : n = 1, 2, \ldots \} \) be a countable independent sequence of events on a probability space \((\Omega, \mathcal{F}, P)\).
- Prove that \( P(\bigcap_{n=1}^{\infty} A_n) = \prod_{n=1}^{\infty} P(A_n). \)
(\textbf{Note:} For a countable sequence of reals, \( \{b_n : n = 1, 2, \ldots \} \), the infinite product \( \prod_{n=1}^{\infty} b_n \) is defined by \( \lim_{n \to \infty} \prod_{k=1}^{n} b_k \), provided this limit exists.)

5. Consider two countable sequences of events, \( \{A_n : n = 1, 2, \ldots \} \) and \( \{B_n : n = 1, 2, \ldots \} \), on the same probability space \((\Omega, \mathcal{F}, P)\). Assume that, for each \( n \), \( A_n \) and \( B_n \) are independent. Moreover, assume that \( A = \lim_{n \to \infty} A_n \) and \( B = \lim_{n \to \infty} B_n \) exist.
- Show that \( A \) and \( B \) are independent.

6. Consider a countable sequence \( \{X_n : n = 1, 2, \ldots \} \) of \( \mathbb{R} \)-valued random variables, defined on a common probability space \((\Omega, \mathcal{F}, P)\), and an increasing function \( G : [0, \infty) \to [0, \infty) \), which satisfies \( \lim_{t \to \infty} \{t^{-1} G(t)\} = \infty \) and \( 0 < \sup_n E\{G(|X_n|)\} < \infty \).
- Prove that \( \{X_n : n = 1, 2, \ldots \} \) is uniformly integrable.