1. Let \( \{A_n : n = 1, 2, \ldots\} \) be a countable sequence of subsets of a sample space \( \Omega \).
   (a) Assume that \( \{A_n : n = 1, 2, \ldots\} \) is an increasing sequence, that is, \( A_n \subseteq A_{n+1} \), for all \( n \geq 1 \).
   Show that \( \lim_{n \to \infty} A_n \) exists, and \( \lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} A_n \).
   (b) Assume that \( \{A_n : n = 1, 2, \ldots\} \) is a decreasing sequence, that is, \( A_{n+1} \subseteq A_n \), for all \( n \geq 1 \).
   Show that \( \lim_{n \to \infty} A_n \) exists, and \( \lim_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} A_n \).

2. By definition, we will say that two events \( A \) and \( B \) in a probability space \( (\Omega, \mathcal{F}, P) \) are equal almost surely (a.s.) if \( P(A \cap B^c) = 0 \) and \( P(B \cap A^c) = 0 \). Prove the following:
   (a) If \( A \) and \( B \) are equal a.s., then \( P(A) = P(B) \).
   (b) \( A \) and \( B \) are equal a.s. \textbf{if and only if} \( P(A \cap B) = \max\{P(A), P(B)\} \).

3. Consider a countable sequence \( \{A_n : n = 1, 2, \ldots\} \) of events in a probability space \( (\Omega, \mathcal{F}, P) \).
   Prove the following inequalities:
   (a) \( \liminf_{n \to \infty} P(A_n) \geq P(\liminf_{n \to \infty} A_n) \)
   (b) \( P(\limsup_{n \to \infty} A_n) \geq \limsup_{n \to \infty} P(A_n) \)

   \textbf{Note}: Based on results for real sequences, we have \( \limsup_{n \to \infty} P(A_n) \geq \liminf_{n \to \infty} P(A_n) \), and therefore, in conjunction with the results in parts (a) and (b) above, we obtain
   \( P(\limsup_{n \to \infty} A_n) \geq \limsup_{n \to \infty} P(A_n) \geq \liminf_{n \to \infty} P(A_n) \geq P(\liminf_{n \to \infty} A_n) \).

4. Consider a measurable space \( (\Omega, \mathcal{F}) \) and a set function \( P: \mathcal{F} \to [0,1] \) which satisfies \( P(\Omega) = 1 \) and \( P(A \cup B) = P(A) + P(B) \) for any \( A \) and \( B \) in \( \mathcal{F} \) with \( A \cap B = \emptyset \). Prove that \( P \) is countably additive (i.e., \( P \) is a probability measure on \( (\Omega, \mathcal{F}) \)) \textbf{if and only if} \( P \) is continuous (i.e., \( P(\lim_{n \to \infty} A_n) = \lim_{n \to \infty} P(A_n) \), for any sequence \( \{A_n : n = 1, 2, \ldots\} \) of sets in \( \mathcal{F} \) for which \( \lim_{n \to \infty} A_n \) exists.)

5. Assuming the usual Borel \( \sigma \)-field on \( R \), prove that any non-decreasing function from \( R \) to \( R \) is measurable.

6. Let \( X \) be a random variable defined on a probability space \( (\Omega, \mathcal{F}, P) \) and taking values on a measurable space \( (\Psi, \mathcal{G}) \). Consider the collection \( \mathcal{A} \) of subsets of \( \Omega \) consisting of \( X^{-1}(B) \) for all \( B \in \mathcal{G} \). Because \( X \) is a random variable, we know that \( \mathcal{A} \subseteq \mathcal{F} \). Show that \( \mathcal{A} \) is a \( \sigma \)-field on \( \Omega \).
   \textbf{(Note):} \( \mathcal{A} \) is referred to as the \( \sigma \)-field generated by the random variable \( X \), and is denoted by \( \sigma(X) \). It is used in the formal definition of independence for random variables.

7. Let \( X \) and \( Y \) be \( R \)-valued random variables defined on the same probability space \( (\Omega, \mathcal{F}, P) \), and consider the subset of \( \Omega \) defined by \( A = \{\omega \in \Omega : X(\omega) \neq Y(\omega)\} \).
   (a) Prove that \( A \) is an event in \( \mathcal{F} \).
   \textbf{(Hint):} Recall the \textit{Archimedean Property} for the set of real numbers, according to which, for any two real numbers \( a \) and \( b \) with \( a < b \), there exists a rational number \( q \) such that \( a < q < b \).
   (b) Assume that \( P(A) = 0 \). Prove that the distributions of \( X \) and \( Y \) are equal, that is, prove that \( P(X^{-1}(B)) = P(Y^{-1}(B)) \) for any Borel subset \( B \) of \( R \).