Problems

8.1 Following the method introduced in class

\[ z = 0 \quad N(z) = a z \]

\[ \omega = \text{constant} \quad k_x = \text{constant} \]

\[ z = h \]

\[ \omega^2 = \frac{N^2 k_x^2}{k^2} \implies k^2 = \frac{N^2 k_x^2}{\omega_0^2} = \frac{a^2 z^2 k_x^2}{\omega_0^2} \]

\[ k_x^2 = -k_x^0 + \frac{a^2 z^2 k_x^2}{\omega_0^2} \]

\[ k_x^2 = k_x^0 \left( \frac{a^2 z^2}{\omega_0^2} - 1 \right) \]

then,

\[ \frac{dz}{dx} = -\frac{k_x^0}{k_x^2} = \mp \frac{k_x^0}{k_x^0 \left( \frac{a^2 z^2}{\omega_0^2} - 1 \right)^{\frac{1}{2}}} \]

\[ \implies \left( \frac{a^2 z^2}{\omega_0^2} - 1 \right)^{\frac{1}{2}} dz = \pm dx \]

let \( u = \frac{a z}{\omega_0} \) then

\[ (u^2 - 1)^{\frac{1}{2}} \frac{\omega_0}{a} \, du = \pm dx \]

\[ \int (u^2 - 1)^{\frac{1}{2}} \, du = \frac{1}{2} u \sqrt{u^2 - 1} - \frac{1}{2} \ln |u + \sqrt{u^2 - 1}| + C \]

\[ \implies \frac{1}{2} \frac{a^2}{\omega_0} \sqrt{\frac{a^2 z^2}{\omega_0^2} - 1} - \frac{1}{2} \ln \left| \frac{a z}{\omega_0} + \sqrt{\frac{a^2 z^2}{\omega_0^2} - 1} \right| + C = \pm \frac{a z}{\omega_0} \]

While it's difficult to invert this into a \( z = f(x) \) function, we can nevertheless plot the ray paths as \( X = g(z) \) as is. 

\[ X = g(z) \]
2.2 Global modes

Consider \( \frac{\partial^2}{\partial t^2} \left( \nabla^2 \phi \right) = -N^2 \frac{\partial^2 \phi}{\partial x^2} \)

1. in rectangular domain with impermeable boundaries:

\[ u=0 \Rightarrow \frac{\partial \phi}{\partial x} = 0 \Rightarrow \phi = \text{constant} \]

\[ w=0 \Rightarrow \frac{\partial \phi}{\partial y} = 0 \Rightarrow \phi = \text{constant} \]

\[ \Rightarrow \text{the edge of the domain is a } \phi = 0 \text{ contour.} \]

2. let \( \phi(x,y,t) = A(x,y)B(t) \)

\[ \Rightarrow \frac{d^2B}{dt^2} \nabla^2 A = -N^2 \frac{\partial^2 A}{\partial x^2} B \]

\[ \Rightarrow \frac{1}{B} \frac{d^2B}{dt^2} = -N^2 \frac{\partial^2 A}{\nabla^2 A \partial x^2} = \text{const} \]

\[ \Rightarrow \text{since we expect oscillations, let this constant be } -\omega^2 \]

\[ \Rightarrow \int \frac{d^2B}{dt^2} = -\omega^2 B \]

\[ \Rightarrow \int \frac{N^2}{\nabla^2 A} \frac{\partial^2 A}{\partial x^2} = -\omega^2 \Rightarrow \omega^2 \nabla^2 A = N^2 \frac{\partial^2 A}{\partial x^2} \]

\[ \text{let's look at the spatial problem:} \]

\[ \omega^2 \left( \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} \right) = N^2 \frac{\partial^2 A}{\partial x^2} \]

\[ \Rightarrow (\omega^2 - n^2) \frac{\partial^2 A}{\partial x^2} + \omega^2 \frac{\partial^2 A}{\partial y^2} = 0 \]

\[ \text{let } A(x,y) = a(x)b(y) \Rightarrow \]
\[(\omega^2 - N^2) \frac{\partial^2 \alpha}{\partial x^2} + \omega^2 \alpha \frac{\partial^2 b}{\partial y^2} = 0\]

\[
\Rightarrow \frac{\omega^2 - N^2}{\alpha} \frac{\partial^2 \alpha}{\partial x^2} + \frac{\omega^2}{b} \frac{\partial^2 b}{\partial y^2} = 0 \Rightarrow \text{each term is constant}
\]

\[
\Rightarrow \begin{cases} 
\frac{\omega^2 - N^2}{\alpha} \frac{\partial^2 \alpha}{\partial x^2} = +k^2 \\
\frac{\omega^2}{b} \frac{\partial^2 b}{\partial y^2} = -k^2 & \text{we need oscillatory in y to fit the homogeneous bcs =)
}
\end{cases}
\]

\[
\text{a. } \frac{\partial^2 b}{\partial y^2} = -\frac{k^2}{\omega^2} b \Rightarrow b = \sin \left( \frac{\pi y}{H} \right) \text{ with } \frac{n^2 \pi^2}{H^2} = \frac{k^2}{\omega^2}
\]

\[
\text{b. } \frac{\partial^2 a}{\partial x^2} = +\frac{k^2}{\omega^2-N^2} a & \text{ we also need } \frac{k^2}{\omega^2-N^2} \text{ to be negative to have oscillation in x to fit the bcs}
\]

\[
\Rightarrow \text{this requires } \omega^2 < N^2
\]

\[
\Rightarrow \frac{\partial^2 a}{\partial x^2} = -\frac{k^2}{N^2-\omega^2} a \Rightarrow a = \sin \left( \frac{m \pi x}{L} \right) \text{ with } \frac{m^2 \pi^2}{L^2} = \frac{k^2}{N^2-\omega^2}
\]

\[
\text{This implies } \frac{m^2 \pi^2}{L^2} = \frac{1}{\frac{N^2-\omega^2}{k^2}} = \frac{1}{\frac{N^2}{k^2} - \frac{H^2}{\pi^2}}
\]

\[
\Rightarrow \frac{N^2}{k^2} = \frac{H^2}{\pi^2} + \frac{L^2}{m^2 \pi^2}
\]

\[
\Rightarrow k^2 = \frac{1}{N^2 \pi^2} \left( \frac{H^2}{n^2} + \frac{L^2}{m^2} \right)
\]
\[ \omega^2 = \frac{k^2 H^2}{n^2 \pi^2} = \frac{N^2 H^2}{h^2 \pi^2} \cdot \frac{1}{N^2 \pi^2 \left( \frac{H^2}{n^2} + \frac{l^2}{m^2} \right)} \]

\[ \omega^2 = \frac{N^2 H^2}{H^2 + l^2 n^2} m^2 \]

This is clearly quite different from the equivalent problem for sound waves, which shouldn't be surprising given that the dispersion relation is so different in the two cases.

\[ \Phi^I + \Phi_R = 0 \text{ on the wall (} x = 0 \text{)} \]

which implies that

\[ \begin{align*}
    \Phi^I &= A e^{ik^x x + i k^y y - i \omega t} \\
    \Phi_R &= A e^{i k^x x + i k^y y - i \omega t}
\end{align*} \]

By the dispersion relation

\[ \omega^2 = \frac{N^2 k^x}{k^x} = \omega^2 = \frac{N^2 k^x}{k^x} \]

\[ \Rightarrow \frac{(k^x)^2}{\varepsilon^x} = \frac{(k^x)^2}{(k^y)^2} \]

\[ \Rightarrow \frac{(k^x)^2}{(k^y)^2 + (k^z)^2} = \frac{(k^x)^2}{(k^y)^2 + (k^z)^2} \Rightarrow \frac{(k^y)^2}{(k^x)^2} = \frac{(k^y)^2}{(k^x)^2} \]
so \((k_x^I)^2 = (k_x^R)^2 \Rightarrow k_x^I = -k_x^R\) (to guarantee outgoing direction)

so finally, we end up with exactly the same reflection properties as pressure waves
Problem 3

Surface waves have dispersion relation

$$\omega^2 = gh \tanh(hk)$$

1. The evolution equations for $$\omega$$ and $$k$$ are

$$\frac{\partial \omega}{\partial t} + c_g \cdot \nabla \omega = \left( \frac{\partial \Omega}{\partial t} \right)_k$$

$$\frac{\partial k}{\partial t} + c_g \cdot \nabla k = \left( - \frac{\partial \Omega}{\partial x} \right)_k$$

$$- \left( \frac{\partial \Omega}{\partial y} \right)_k$$

where $$\Omega = \omega(k) = \sqrt{gh \tanh(hk)}$$

$$c_g = \frac{\partial \omega}{\partial k} = \frac{g \tanh(hk) + hkg \cosh^2(hk)}{2 \sqrt{gh \tanh(hk)}}$$

If $$h$$ is a function of time $$\to$$ $$\omega$$ is not conserved

If $$h$$ is a function of $$x, y$$ $$\to$$ $$k$$ is not conserved.

2. If $$h = s y$$ then

$$\omega$$ is conserved

$$k_x$$ is conserved

as the wave approaches the beach, $$h$$ becomes small then $$\tanh(hk) \approx hk$$

then $$\omega = \sqrt{gkhk} \approx k \sqrt{gsy}$$

$$\Rightarrow \quad \omega = \left( k_x^2 + k_y^2 \right)^{1/2} \sqrt{gsy}$$

$$\Rightarrow \quad \frac{\omega^2}{gsy} = k_x^2 + k_y^2$$
\[ k_y^2 = \frac{\omega^2}{g s y} - k_x^2 \]

as \( y \to 0 \) then \( k_y \to \infty \)

In other words, 

\[ \text{As } y \to 0, \text{ then } k_y \to \infty \]

k turns to become \perp to the beach

the wave crests that are \perp to \( k \) turn to be \parallel to the beach!