Application to non-periodic systems

As discussed in the introduction, the power of the Lighthill / renormalization methods is that they can also be used for non-periodic systems (by contrast with the Lindstedt–Poincaré technique which cannot).

We now see an example of how that may work.

Consider the equation

\[ (x + \varepsilon f) \frac{df}{dx} + f = 0 \quad 0 < \varepsilon \ll 1 \]
\[ f(1) = 1. \]

Before we proceed with the perturbation expansion, note that an exact solution to this equation actually exists. Let's see what it is, to understand what is about to happen.

\((x + \varepsilon f) \frac{df}{dx} + f = 0 \) can be rewritten as

\((1 + \varepsilon f) \frac{df}{dx} + \frac{f}{x} = 0 \) this is now a homogeneous form

To find a solution, let \( g = \frac{f}{x} \) then \( f = xg \) so

\[ \frac{df}{dx} = x \frac{dg}{dx} + g \]

\[ \Rightarrow (1 + \varepsilon g) \left( g + x \frac{dg}{dx} \right) + g = 0 \Rightarrow x \frac{dg}{dx} = - \frac{g}{1 + \varepsilon g} - g \]

\[ = - \frac{g + (1 + \varepsilon g)g}{1 + \varepsilon g} \]

so

\[ \frac{dg}{g} \frac{(1 + \varepsilon g)}{2g + \varepsilon g^2} = - \frac{dx}{x} \]

\[ \Rightarrow \frac{1}{2} \ln \left( g + \frac{\varepsilon g^2}{2} \right) = - \ln |x| + \text{const} \]
\[ g + \varepsilon \frac{g^2}{2} = \frac{k}{x^2} \]
\[ \frac{\varepsilon g^2}{2} + g - \frac{k}{x^2} = 0 \]
\[ g = \frac{-1 \pm \sqrt{1 + \frac{2\varepsilon KE}{x^2}}}{\varepsilon} \]
\[ f(x) = \frac{-x \pm \sqrt{x^2 + 2\varepsilon KE}}{\varepsilon} \]

To have \( f(0) = 1 \),
\[ 1 = \frac{-1 \pm \sqrt{1 + 2\varepsilon KE}}{\varepsilon} \]
\rightarrow this cannot happen for the - solution (assuming \( \varepsilon > 0 \)), so we must keep only the + solution. Then, we need
\[ \varepsilon = -1 + \sqrt{1 + 2\varepsilon KE} \Rightarrow (\varepsilon + 1)^2 = 1 + 2\varepsilon KE \]
\[ \Rightarrow k = \frac{(\varepsilon + 1)^2 - 1}{2\varepsilon} = \frac{\varepsilon^2 + 2\varepsilon}{2\varepsilon} = \frac{\varepsilon + 2}{2} \]
And finally,
\[ f(x) = \frac{-x + \sqrt{x^2 + \varepsilon(E+2)}}{\varepsilon} \]

Note now: for \( x = 0 \), \( f(0) \) is well-defined and equal to \( f(0) = \sqrt{\frac{E+2}{\varepsilon}} \).

- for \( x \to \pm \infty \), \( f(x) \approx \frac{-x + |x|\sqrt{1 + \frac{\varepsilon(E+2)}{x^2}}}{\varepsilon} \)

\[ \approx \frac{-x + |x|\left(1 + \frac{\varepsilon(E+2)}{2x^2}\right)}{\varepsilon} \frac{1}{\varepsilon} \]
\[ = -\frac{x}{\varepsilon} + \frac{|x|}{\varepsilon} + \frac{|x|(E+2)}{2x^2} + \ldots \]

So if \( x > 0 \) then \( f(x) \approx \frac{E+2}{2x} + \ldots \)

If \( x < 0 \) then \( f(x) \approx -\frac{2x}{\varepsilon} - \frac{E+2}{2x} \)
\[ \frac{d^2 f}{dx^2} - \frac{f}{x} = 0 \]

As \( x \to 0^+ \), \( \frac{d^2 f}{dx^2} \to \infty \), \( f \to 0 \) is singular.

To 1st order, \( f(x) = 0 \). Let's now see what a standard perturbation technique yields. We begin by naively assuming an expansion of the form \( f(x, \epsilon) = f_0(x) + \epsilon f_1(x) + \cdots \) and if \( \epsilon f_0(x) + \epsilon f_1(x) \ll f_0(x) \), then

\[ f_1(x) = \frac{1}{x} \frac{d^2 f}{dx^2} - \frac{f_0(x)}{x} = 0 \]

\[ \frac{df_1}{dx} = \frac{1}{x} \frac{d}{dx} \left( \frac{d^2 f_0}{dx^2} - \frac{f_0(x)}{x} \right) \]

\[ \frac{df_1}{dx} = \frac{1}{x} \frac{d}{dx} \left( \frac{d^2 f_0}{dx^2} \right) \]

\[ f_1(x) = \frac{1}{x} \frac{d}{dx} \left( \frac{d^2 f_0}{dx^2} \right) \]

\[ \frac{df_1}{dx} = \frac{1}{x} \frac{d}{dx} \left( \frac{d^2 f_0}{dx^2} \right) \]

\[ f_1(x) = \frac{1}{x} \frac{d}{dx} \left( \frac{d^2 f_0}{dx^2} \right) \]

Check: \( f(0) = 0 \), \( \frac{df}{dx} = 0 \) as \( x \to 0^+ \). Hence the solution is singular (and in fact,\( \frac{df}{dx} \) tends to the vertical axis).
\[ x f_1' + f_1 = - f_0 f_0' = \frac{1}{x^3} \]
\[ \Rightarrow (xf_1)' = \frac{1}{x^3} \Rightarrow xf_1 = - \frac{1}{2x^2} + K_1 \]
\[ \text{for } f_1(1) = 0 \Rightarrow 0 = - \frac{1}{2} + K_1 \Rightarrow K_1 = \frac{1}{2} \]
So \[ f_1(x) = \frac{1}{2x} \left( 1 - \frac{1}{x^2} \right) \]

\[ \Rightarrow \text{To 2nd order,} \]
\[ xf_0' + f_a + f_f' + f_0f' \quad \text{with } f_2(1) = 0 \]
\[ \Rightarrow (f_2 x)' = -(f_1 f_0)' \]
\[ \Rightarrow f_2 x = - f_0 f_0 + K_2 = - \frac{1}{2x^2} \left( 1 - \frac{1}{x^2} \right) + K_2 \]
\[ f_2(1) = 0 \Rightarrow \quad \text{so } f_2(x) = - \frac{1}{2x^3} \left( 1 - \frac{1}{x^2} \right) \]

So finally, \[ f(x) = \frac{1}{x} + \frac{e}{2x} \left( 1 - \frac{1}{x^2} \right) - \frac{e^2}{2x^3} \left( 1 - \frac{1}{x^2} \right)^2 + \cdots \]
\[ \Rightarrow \frac{1}{x} \left[ 1 + \frac{e}{2} \left( 1 - \frac{1}{x^2} \right) - \frac{e^2}{2} \left( 1 - \frac{1}{x^2} \right)^2 + \cdots \right] \]

It's quite clear that this is a non-uniform expansion as \( x \to 0 \). Indeed, the remainders cannot be bounded in any \( x \)-independent way, and the singularity increases with the order of expansion: \[ f_0 \sim o\left( \frac{1}{x} \right) \quad f_1 \sim o\left( \frac{1}{x^3} \right) \quad f_2 \sim o\left( \frac{1}{x^5} \right) \]

Let's now try to apply renormalization on this non-uniform expansion to try to see if we can remove the problem.
Let \( x = s + \varepsilon a_1(s) + \varepsilon^2 a_2(s) + \cdots \)
Then

\[ \phi(s) \approx \frac{1}{s + \varepsilon a_1(s) + \ldots} \left[ 1 + \frac{\varepsilon \sqrt{2}}{2} \left( 1 - \frac{1}{(s + \varepsilon a_1(s) + \ldots)^2} \right) \right. \\
\left. - \frac{\varepsilon^2}{2(s + a_1(s) + \ldots)^2} \left( 1 - \frac{1}{(s + \varepsilon a_1(s) + \ldots)^2} \right) \right] \]

\[ \approx \frac{1}{s \left[ 1 + \frac{\varepsilon}{s} a_1(s) + \ldots \right]} \left[ \ldots \right] \]

\[ \approx \frac{1}{s} \left( 1 - \frac{\varepsilon}{s} a_1(s) + \frac{\varepsilon^2}{s^2} a_1^2(s) - \frac{\varepsilon^2}{s} a_2(s) \right) \left[ \ldots \right] \]

to lowest orders in \( \varepsilon \):

\[ \phi(s) \approx \frac{1}{s} + \varepsilon \left[ - \frac{a_1(s)}{s^2} + \frac{1}{s} \left( \frac{1}{s} \left( 1 - \frac{1}{s^2} \right) \right) \right] \\
+ \varepsilon^2 \left[ \frac{a_1^2(s)}{s^3} - \frac{a_2}{s^2} - \frac{a_1}{s^2} \left( \frac{1}{s} \left( 1 - \frac{1}{s^2} \right) \right) \right. \\
- \frac{1}{s} \left( 1 - \frac{1}{s^2} \right) \left[ \frac{1}{s} \left( 1 - \frac{1}{s^2} \right) \right) \right] \\
\approx \frac{1}{s} + \varepsilon \left[ - \frac{a_1(s)}{s^2} + \frac{1}{2s} \left( 1 - \frac{1}{s^2} \right) \right] \\
+ \varepsilon^2 \left[ - \frac{a_2}{s^2} + \frac{a_1^2}{s^3} - \frac{a_1}{2s^2} + \frac{3a_1}{2s^4} - \frac{1}{2s^3} \left( 1 - \frac{1}{s^2} \right) \right]. \]

The fundamental singularity is in \( \frac{1}{s} \), we cannot hope to remove it; it's part of the solution. However, we can then choose all of the \( a_1, a_2, \ldots \) functions so that the next orders are no more singular than this first one.

* To the order \( \varepsilon \) equation we want to eliminate the terms in \( \frac{1}{s^2} \) and \( \frac{1}{s^3} \). Let's choose

\[ \frac{a_1}{s^2} = - \frac{1}{2s^3} \quad \Rightarrow \quad a_1 = - \frac{1}{2s} \]
In the $O(e^2)$ equation, there are only terms in $\frac{1}{s^2}$, $\frac{1}{s^3}$, $\frac{1}{s^4}$ and $\frac{1}{s^5} \Rightarrow$ we need to eliminate all of them!

$\Rightarrow \quad \frac{a_2}{s^2} = \frac{a_1^2}{s^3} - \frac{a_1}{2s^2} + \frac{3}{2} \frac{a_1}{s^4} - \frac{1}{2s^3} \left( 1 - \frac{1}{s^2} \right)$

$= \frac{1}{4s^3} - \frac{1}{2s^3} = -\frac{1}{4s^3}$

$\Rightarrow \quad a_2 = -\frac{1}{4s}$

So finally,

$$\begin{align*}
\left\{ \begin{array}{l}
x = s - \frac{e}{2s} - \frac{e^2}{4s} + o(e^3) \\
f(s) = \frac{1}{s} + e \left( \frac{1}{2s} \right) + o(e^3)
\end{array} \right.
\end{align*}$$

To find out $s$ in terms of $x$, let's solve for it:

$$s^2 - xs - \frac{e}{2} \left( 1 + \frac{e}{2} \right) = 0$$

$$s = \frac{x \pm \sqrt{x^2 + 2e \left( 1 + \frac{e}{2} \right)}}{2} + \text{h.o.t.} \quad \text{(select + root to have } s > 0 \text{ when } x > 0)$$

$$\Rightarrow \quad f(x) \approx \left( \frac{1 + \frac{e}{2}}{2} \right) \frac{2}{x + \sqrt{x^2 + 2e \left( 1 + \frac{e}{2} \right)}}$$

Check we now have

* $f(0) = \sqrt{\frac{2}{e}} \cdot \sqrt{1 + \frac{e}{2}}$ to lowest order in $e$,

  which is no longer singular but instead fairly close to the right solution

* When $x \to -\infty$, $|x| = -x$

  $$f(x) \approx \left( 1 + \frac{e}{2} \right) \frac{2}{x} \cdot \frac{1}{1 - \sqrt{1 + \frac{2e(1 + e/2)}{x^2}}}$$

  $\approx \left( 1 + \frac{e}{2} \right) \frac{2}{x} \cdot \frac{1}{1 - 1 - e(1 + e/2)} \approx -\frac{2x}{e}$ as required
We therefore see that
\[ f(x) = \frac{2(1 + \frac{\varepsilon}{2})}{x + \sqrt{x^2 + 2\varepsilon(1 + \frac{\varepsilon}{2})}} \]
is indeed a uniform expansion for \( f(x) \), valid in all limits of interest.

Note: The choices of \( a_1(s) \), \( a_2(s) \), etc are not unique: you can always add a term that is of the same order of singularity as the one you need to keep without changing the uniformity of the expansion. The trick, therefore, is to choose these functions to make the solution as simple as possible.

(3) The failure of the method of strained coordinates

It is important to note that the various method learned in this chapter do not necessarily always work. One should therefore try to apply them, and if they fail, then move to a different set of tools.

A common example for the failure of the method of strained coordinates arises in periodic systems where the nonlinearity/perturbation induces a modulation in the amplitude of oscillation that proceeds on a different timescale from the oscillation period. This occurs, for instance, in the van der Pol oscillator:
\[ \frac{d^2 f}{dt^2} + f = \varepsilon (1 - f^2) \frac{df}{dt}. \]
let's try to apply the techniques learned here, with 24. \( f'(t) \) satisfying \( f(0) = 1, \frac{df}{dt}(0) = 0 \).

First assume \( f'(t) = f_0(t) + 2f_1(t) + \varepsilon^2 f_2(t) \)

\[ \Rightarrow f_0'' + f_0 = 0 \quad f_0(0) = 1 \quad \frac{df_0}{dt}(0) = 0 \]

\[ \Rightarrow f_0(t) = \cos t \]

\[ \Rightarrow f_1'' + f_1 = (1 - f_0^2) \frac{df_0}{dt} = (1 - \cos^2 t)(-\sin t) \]

\[ = -\sin^3 t = \frac{1}{4} \sin^3 t - \frac{3}{4} \sin t \]

\[ \Rightarrow f_1(t) = 4 \cos t + B \sin t + k_1 t \cos t + k_2 \sin 3t \]

\( k_1, k_2 \) satisfy

\[ -9k_2 + k_2 = \frac{1}{4} \quad \Rightarrow \quad k_2 = -\frac{1}{32} \]

\[ -2k_1 = -\frac{3}{4} \quad \Rightarrow \quad k_1 = \frac{3}{8} \]

To satisfy the IC \( f_1(0) = 0, \frac{df_1}{dt}(0) = 0 \), we then have to have

\[ \begin{cases} A = 0 \\ B + \frac{3}{8} \cdot \frac{3}{32} = 0 \end{cases} \quad \Rightarrow \quad B = -\frac{9}{32} \]

so \( f(t) = -\frac{9}{32} \sin t + \frac{3}{8} t \cos t - \frac{1}{32} \sin 3t \)

and so \( f(t) = \cos t + \varepsilon \left( \frac{3}{8} t \cos t - \frac{9}{32} \sin t - \frac{1}{32} \sin 3t \right) \)

so far, everything seems to be as usual. Let's try to get a uniform 1-form expansion for \( f'(t) \), using renormalization
\[ t = e^{2 \omega_1 z} + e^{2 \omega_2 z} + \cdots \]

\[ f(\omega) = \omega_0 + e^{2 \omega_1 z} + e^{2 \omega_2 z} + \cdots + e^{2 \omega_3 z} \left( \frac{3}{8} \right) \]

\[ \frac{1}{2} \left( 2 \omega_2 \omega_1 + 2 \omega_1 + 2 \omega_2 \right) \left( \omega_2 - \frac{9}{32} \right) \sin \left( \frac{1}{2} \, 2 \omega_2 \omega_1 z \right) \]

\[ = \omega_0 e^{2 \omega_1 z} (z + 2 \omega_1) \left( \frac{3}{8} \right) \sin \left( \frac{1}{2} \, 2 \omega_2 \omega_1 z \right) \]

To remove the secular term, we would need

\[ \omega_1 \omega_2 = \frac{3}{8} \frac{2 \omega_1}{\sin \omega_1 \omega_2} \]

The problem is that this function is not defined for all values of \( z \to \) it is singular whenever \( z = \pi ).

Also, it is multiply defined (not "one-to-one") so when writing \( t = e^{2 \omega_1 z} \), \( z \) is not invertible for all \( t \).

This example is fairly typical of the failure of the shrunked coordinate technique. In the next chapter, we will now study a very different approach which can be used to understand the dynamics of the van der Pol oscillator.