CHAPTER 6: The method of dominant balance
2 The WKB approximation

I Regular, singular, and regular-singular points: Frobenius series

In all that follows (except where explicitly mentioned), we now consider the general ODE

$$\frac{d^2 f}{dx^2} + p(x) \frac{df}{dx} + q(x)f = 0$$

Subject to some BCs.

1 Definitions

Consider a point \( x_0 \) in the interval between the BCs.

- If \( p(x_0) \) and \( q(x_0) \) are finite (i.e., not \( \pm \infty \))
  then \( x_0 \) is a regular point; otherwise \( x_0 \) is a singular point.

- If \( x_0 \) is singular, but both
  \[
  \lim_{x \to x_0} (x-x_0) p(x) \quad \text{and} \quad \lim_{x \to x_0} (x-x_0)^2 q(x)
  \]
  are finite (as above) then \( x_0 \) is a regular singular point; otherwise \( x_0 \) is called an irregular singular point.

Example 1. The equation \( \frac{d^2 f}{dx^2} = xf \) is called an Airy equation

\[
\begin{cases}
p(x) = 0 \quad \text{for all points} \\
q(x) = x
\end{cases}
\]

\( \Rightarrow \) all points are regular except \( x \to \pm \infty \).
To find out the behavior near $\infty$, let $s = \frac{1}{x}$ and study the behavior as $s \to 0$

\[
\frac{d}{dx} = \frac{ds}{dx} \frac{d}{ds} = -\frac{1}{x^2} \frac{d}{ds} = -s^2 \frac{d}{ds}
\]

\[
\frac{d^2}{dx^2} = -s^2 \frac{d}{ds} \left( -s^2 \frac{d}{ds} \right) = s^2 \frac{d}{ds} \left( s^2 \frac{d}{ds} \right)
\]

\[
= s^4 \frac{d^2f}{ds^2} + 2s^3 \frac{df}{ds} - \frac{1}{s^2} f = 0
\]

\[
= \frac{d^2f}{ds^2} + \frac{2}{s} \frac{df}{ds} - s^2 f = 0
\]

$p(s) = \frac{2}{s}$ and $q(s) = -\frac{1}{s^2}$

$\lim_{s \to 0} p(s)$ and $\lim_{s \to 0} q(s)$ are infinite

$\lim_{s \to 0} sp(s) = 2$ but $\lim_{s \to 0} q(s)$ is infinite

$\Rightarrow$ The point at $\infty$ is an irregular-singular pt.

for the Any function

**Example 2:** Bessel Equation

\[x^2 \frac{d^2f}{dx^2} + x \frac{df}{dx} + x^2 f = 0\]

\[\Rightarrow \frac{d^2f}{dx^2} + \frac{1}{x} \frac{df}{dx} + f = 0\]

all points are regular except $x = 0$. ($p(x) = \frac{1}{x}$, $q(x) = 1$)

$\lim_{x \to 0} xp(x) = 1$  \hspace{1cm} $\lim_{x \to 0} x^2q(x) = 0$ both finite

$\Rightarrow$ $0$ is a regular-singular point of the Bessel Equation.

**Example 3**

\[\frac{d^2f}{dx^2} + \frac{f}{x^4} = 0\]

By contrast, $x = 0$ is irregular-singular. Near $x = \infty$, however this becomes

\[\frac{d^2f}{ds^2} + \frac{2}{s} \frac{df}{ds} + f = 0 \Rightarrow s = 0 (x = \infty) \text{ is regular-singular.} \]
Theorem:
Suppose $x = 0$ is a regular point or a regular-singular point. Then at least one solution of the ODE
$$\frac{d^2f}{dx^2} + p(x)\frac{df}{dx} + q(x) = 0$$
has the form
$$f_1(x) = x^\alpha \sum_{k=0}^\infty a_k x^k$$
where $a_0 \neq 0$.

This solution is called the Frobenius series.

Notes:
- In general, both solutions of the ODE have this form. However, in some circumstances (see below) the second solution instead has the form
  $$f_2(x) = b \ln |x| f_1(x) + x^\alpha \sum_{k=0}^\infty b_k x^k,$$
  where $b$ is constant $\neq 0$.
  (Note that with $b = 0$ we recover the first form).
- For an irregular-singular point, this does not work and a different method must be used (see later).

However, one can also recast the equation in terms of $s = \frac{1}{x}$, and study the ODE in $s$ near $s = 0$ to get a Frobenius series in $s$ if $s = 0$ is a regular or regular-singular pt.

- While the infinite series usually converges, the truncated series accuracy depends on the distance to the closest irregular-singular pt.
- We can also construct series near $x_0$ where $x_0 \neq$
Example 1  Spherical Bessel functions of order 0.

The equation for this function is \( \frac{d^2f}{dx^2} + \frac{2}{x} \frac{df}{dx} + f = 0 \)
we saw before that it is regular everywhere except \( x = 0 \); \( x = 0 \) is a regular-singular pt.

We then have \( f(x) = \sum_{k=0}^{\infty} a_k x^{\alpha + k} \)

\[ \frac{df}{dx} = \sum_{k=0}^{\infty} a_k (\alpha + k) x^{\alpha + k - 1} \]
\[ \frac{d^2f}{dx^2} = \sum_{k=0}^{\infty} a_k (\alpha + k) (\alpha + k - 1) x^{\alpha + k - 2} \]

so the equation becomes:

\[ \sum_{k=0}^{\infty} a_k (\alpha + k) (\alpha + k - 1) x^{\alpha + k - 2} + \alpha \sum_{k=0}^{\infty} a_k (\alpha + k) x^{\alpha + k - 2} + \sum_{k=0}^{\infty} a_k x^{\alpha + k} = 0 \]

Dividing by \( x^{\alpha} \), & equating orders of this function of \( x \), we get

To order \( x^{-2} \), \( a_0 \alpha (\alpha - 1) + 2a_0 \alpha = 0 \)

To order \( x^{-1} \): \( a_1 (\alpha + 1) \alpha + 2a_1 (\alpha + 1) = 0 \)

To order \( x^n \) (\( n \geq 0 \)) \( a_{n+2} (\alpha + n + 2)(\alpha + n + 1) + 2a_{n+2} (\alpha + n + 2) + a_n = 0 \)

From the lowest order we get \( \alpha (\alpha + 1) = 0 \Rightarrow \)

\( \alpha = 0 \) or \( \alpha = -1 \)

\( \Rightarrow \) This yields the 2 fundamental solutions.

Let's study them separately:
\( a = 0 \): \( a_0 \) is arbitrary (that's our integration constant)

- The order \( x^{-1} \) yields: \( 2a_1 = 0 \Rightarrow a_1 = 0 \)
- The next orders yield:

\[
a_{n+2} \left( \alpha + n + 2 \right) \left( \alpha + n + 3 \right) = -a_n \quad \text{(in general)}
\]

\[
a_{n+2} = -\frac{a_n}{(n+2)(n+3)}
\]

so:

\[
a_2 = -\frac{a_0}{3!} \quad a_3 = 0 \quad a_4 = -\frac{a_2}{4 \times 5} = \frac{a_0}{5!}
\]

... etc

So the first fundamental solution is:

\[
f_1(x) = a_0 \left( -\frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \right)
\]

\[
= \frac{a_0}{x} \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right) = \frac{a_0 \sin x}{x}
\]

\( \alpha = -1 \): The order \( x^{-1} \) yields nothing \( \Rightarrow a_1 \) arbitrary as well

- Next orders:

\[
a_{n+2} = -\frac{a_n}{(\alpha + n + 2)(\alpha + n + 3)} = -\frac{a_n}{(n+1)(n+2)}
\]

so:

\[
a_2 = -\frac{a_0}{2} \quad a_3 = -\frac{a_1}{3!}
\]

\[
a_4 = -\frac{a_2}{3 \times 4} \quad a_5 = -\frac{a_3}{4 \times 5} = \frac{a_1}{5!}
\]

... etc...

\[
f_2(x) = \frac{a_0}{x} \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right) \leftarrow \text{new solution}
\]

\[
+ \frac{a_1}{x} \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right) \leftarrow \text{old solution}
\]

\[
= \frac{a_0 \cos x}{x} + \frac{a_1 \sin x}{x}
\]

\( \Rightarrow \) we see that the spherical Bessel eq. has 2 fundamental solutions: \( j_0(x) = \frac{\sin x}{x} \), \( y_0(x) = \frac{\cos x}{x} \)
Example 2 A more general case (slightly harder problem)

Consider the equation \( \frac{d^2 f}{dx^2} + p(x) \frac{df}{dx} + q(x)f = 0 \)

where both \( p(0) \) and \( q(0) \) are finite and non-zero.

- \( x = 0 \) is a regular point, so we can try the Frobenius series:
  \[ f(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha} \]

as before:

\[
\begin{align*}
\frac{df}{dx} &= \sum_{k=0}^{\infty} a_k (k+\alpha)x^{k+\alpha-1} \\
\frac{d^2 f}{dx^2} &= \sum_{k=0}^{\infty} a_k (k+\alpha)(k+\alpha-1)x^{k+\alpha-2}
\end{align*}
\]

\[
= \sum_{k=0}^{\infty} a_k (k+\alpha)(k+\alpha-1)x^{k+\alpha-2}
\]

\[ + \ v(x)\sum_{k=0}^{\infty} a_k (k+\alpha)x^{k+\alpha-1} + q(x)\sum_{k=0}^{\infty} a_k x^{k+\alpha} = 0
\]

Since \( x = 0 \) is regular, we can Taylor expand \( p \) and \( q \) without difficulty. This yields:

\[ p(x) = p(0) + xp'(0) + \frac{x^2}{2} p''(0) + \cdots = \sum_{j=0} \frac{p(j)(0)x^j}{j!} \]

and similarly for \( q \).

Plugging this in, and dividing by \( x^\alpha \), we get:

\[
\sum_{k=0}^{\infty} a_k (k+\alpha)(k+\alpha-1)x^{k-2}
\]

\[ + \ \sum_{j,k=0}^{\infty} \frac{p(j)(0)}{j!} a_k x^{k+j-1} + \sum_{j,k=0}^{\infty} q(j)(0) a_k x^{k+j} = 0
\]

Equating powers of \( x \), we then get:

\[ 0(x^{-2}) : \quad a_0 \alpha (\alpha-1) = 0 \]

\[ 0(x^{-1}) : \quad a_1 (\alpha+1) + \alpha p(0) a_0 = 0 \]
\[ 0 \left( x^2 \right) : \quad a_2 \left( 2 + x \right) \left( 1 + x \right) + (1 + x) p(0)a_1 + x p'(0) a_0 + q(0) a_0 = 0 \]

\[ 0 \left( x^n \right) : \quad a_{n+2} \left( n + 2 + x \right) \left( n + 1 + x \right) \]
\[ + \sum_{k=0}^{n+1} \frac{(\alpha + k) p^{(n+1-k)}(0) a_k}{(n+1-k)!} + \sum_{k=0}^{n} \frac{q^{(n-k)}(0) a_k}{(n-k)!} = 0 \]

From the lowest order we get (since \( a_0 \neq 0 \) by definition)
\[ \alpha = 0 \quad \text{or} \quad \alpha = 1 \quad \Rightarrow \quad \text{This gives us our 2 fundamental solutions.} \]

In the case \( \alpha = 0 \):
- \( a_0 \) is arbitrary (that's the integration constant).
- The \( 0(x^1) \) equation doesn't tell us anything.
- The \( 0(x^0) \) equation states
  \[ 2a_2 + p(0)a_1 + q(0)a_0 = 0 \]
  \[ \Rightarrow \text{choose (arbitrarily) } a_1 = 0 \Rightarrow \]
  \[ a_2 = -\frac{q(0)}{2} a_0 \]
- The \( 0(x^n) \) equation gives us the recurrence to get all solutions:
  \[ a_{n+2} \left( n + 2 \right) \left( n + 1 \right) = -\sum_{k=0}^{n+1} \frac{k p^{(n+1-k)}(0) a_k}{(n+1-k)!} - \sum_{k=0}^{n} \frac{q^{(n-k)}(0) a_k}{(n-k)!} \]
  \[ \forall n \geq 0 \]

We can then do the same for \( \alpha = 1 \):
- The \( 0(x^{1-1}) \) equation states that:
  \[ 2a_1 + p(0)a_0 = 0 \Rightarrow a_1 = -\frac{p(0)a_0}{2} \]
- The \( 0(x^0) \) equation states that:
  \[ 6a_2 + 2p(0)a_1 + p'(0)a_0 + q(0)a_0 = 0 \]
  \[ \Rightarrow \text{yields } a_2 \quad \text{etc} \ldots \]
Note: were we allowed to pick $a_1 = 0$ in the first case?

As it turns out, it's exactly the same issue as for the spherical Bessel function. Suppose we had not picked $a_1 = 0$.

Then, in the first solution we would get

$$a_2 = -\frac{9(0)}{2} a_o - \frac{p(0)}{2} a_1$$

etc.

$\Rightarrow$ The lowest order of $f(x)$ can be written as

$$f_1(x) = a_0 + a_1 x + \left(-\frac{9(0)}{2} a_o - \frac{p(0)}{2} a_1\right)x^2 + \ldots$$

Meanwhile, the second solution was

$$f_2(x) = x \left[ \tilde{a}_0 - \frac{p(0)}{2} \tilde{a}_0 x + \ldots \right]$$

We see that $f_2(x)$ is in fact contained in the $f_1(x)$ solution (in the terms in $a_1$.)

$\Rightarrow$ either we set $a_1$ to 0 & then we have to find both solutions

$\Rightarrow$ or we don't set $a_1$ to 0 & then "$f(x)$" actually contains both fundamental solutions.
Example 3: (An even harder problem): The Bessel Equation: $x^2 f''(x) + x f'(x) + f = 0 \rightarrow$ has a regular singular point.

We assume $f(x) = \sum_{k=0}^{\infty} a_k x^{\alpha+k}$

$\Rightarrow$ expand $\frac{df}{dx}$, $\frac{d^2 f}{dx^2}$ as usual $\rightarrow$

$$\sum_{k=0}^{\infty} a_k (\alpha+k)(\alpha+k-1)x^{\alpha+k-2} + \sum_{k=0}^{\infty} a_k x^{\alpha+k} = 0 \rightarrow$$

Divide by $x^{\alpha}$

2. Equate powers of $x$, as usual $\Rightarrow$

$0(x^{-2})$: $a_0 (\alpha-1)\alpha + a_0 \alpha = 0 \Rightarrow a_0 \alpha^2 = 0$

$0(x^{-1})$: $a_1 (\alpha+1)\alpha + a_1 (\alpha+1) = 0 \Rightarrow a_1 (\alpha+1)^2 = 0$

$0(x^n)$: $a_{n+2} (\alpha+n+2) (\alpha+n+1) + a_{n+2} (\alpha+n+2) + a_n = 0$

The lowest order equation suggests that $\alpha = 0$,

but this time it's a double root (so we will only get one solution from this).

Let's have $\alpha = 0$: $\Rightarrow a_0$ arbitrary.

$\cdot$ from $0(x^{-1}) \Rightarrow a_1 = 0$

$\cdot$ from $0(x^n) \Rightarrow a_{n+2} = -\frac{a_n}{(\alpha+n+2)^2} = -\frac{a_n}{(n+2)^2}$

So

$$f(x) = a_0 - \frac{a_0}{4} x^2 + \frac{a_0 x^4}{64} - \ldots$$

$$= a_0 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \ldots\right)$$

To get the next solution, we remember that it could take the form

$$f_2(x) = b \ln(1+x) f_1(x) + \sum_{k=0}^{\infty} b_k x^{\alpha_2+k}$$
Note that since $a_0$ is arbitrary, we can also arbitrarily choose $b_0 = 1$ (and fold $a_1$ into $a_0$)

$$f_2(x) = \ln|1| f_1(x) + \sum_{k=0}^{\infty} b_k x^{d_2+k}$$

$$\frac{d f_2}{dx} = \frac{1}{x} f_1 + \ln|1| \frac{d f_1}{dx} + \sum_{k=0}^{\infty} b_k (d_2+k) x^{d_2+k-1}$$

$$\frac{d^2 f_2}{dx^2} = -\frac{1}{x^2} f_1 + \frac{2}{x} \frac{d f_1}{dx} + \ln|1| \frac{d^2 f_1}{dx^2} + \sum_{k=0}^{\infty} b_k (d_2+k)(d_2+k-1) x^{d_2+k-2}$$

Plugging this in, we then get (collecting similar terms):

$$\ln|1| \left[ \frac{d^2 f_1}{dx^2} + \frac{2}{x} \frac{d f_1}{dx} + \frac{1}{x^2} f_1 \right] + \frac{2}{x} \frac{d f_1}{dx} - \frac{1}{x^2} f_1 + \frac{1}{x^2} f_1$$

$$+ \sum_{k=0}^{\infty} b_k (d_2+k)(d_2+k-1) x^{d_2+k-2} + \sum_{k=0}^{\infty} b_k (d_2+k) x^{d_2+k-2} + \sum_{k=0}^{\infty} b_k x^{d_2+k} = 0$$

The second line is exactly what we had earlier for the $a_1$ and $a_k$ variables. It would give the same solution were it not for the leftover term on the first line, which has $\frac{2}{x} \frac{d f_1}{dx} = 2 \sum_{k=1}^{\infty} a_k k x^{k-2}$

$$= \text{Now, dividing by } d_2, \text{ we are left with:}$$

$$\sum_{k=0}^{\infty} b_k (d_2+k)(d_2+k-1) x^{k-2} + \sum_{k=0}^{\infty} b_k (d_2+k) x^{k-2}$$

$$+ \sum_{k=0}^{\infty} b_k x^k + 2 \sum_{k=1}^{\infty} a_k k x^{k-2} = 0$$

Expanding term by term & keeping only the lowest few in each case we get:

$$b_0 d_2 (d_2-1) x^{-2} + b_1 (d_2+1) d_2 x^{-1} + \ldots + b_0 d_2 x^{-2} + b_1 (d_2+1) x^{-1} + \ldots = 0$$
\[ + b_0 + a_1 x + \ldots + \frac{2a_1 x^{-1 - \alpha_2}}{\alpha_2} + 2a_2 x^{-\alpha_2} + \ldots = 0 \]

We see that there are a few possibilities.

- \[ \alpha_2 < -2 \quad \text{in which case we would need} \quad a_2 = 0 \quad \text{(} \alpha_2 > 2 \text{)} \quad \text{inconsistent since} \quad a_0 \neq 0 \]

- \[ \alpha_2 = -2 \quad \text{(} \alpha_2 < 2 \text{)} \]

(\(\text{to } O(x^{-1})\)):
\[ b_0 a_2 (\alpha_2 - 1) + b_0 a_2 + 4a_2 = 0 \]
\[ \Rightarrow b_0 a_2^2 + 4a_2 = 0 \quad \Rightarrow \text{thus gives} \quad b_0 = -\frac{4a_2}{a_2^2} \]

(\(\text{to } O(x^{-1})\)):
\[ b_1 (\alpha_2 + 1)a_2 + b_1 (\alpha_2 + 1) + 6a_3 = 0 \quad \Rightarrow \text{thus gives} \quad b_1 = \frac{6a_3}{\alpha_2 + 1} \]

This works & gives a unique solution.

- \[ \alpha_2 > -2 \quad \text{in which case the lowest order is} \]

(\(\text{to } O(x^{-1})\)):
\[ b_0 a_2 (\alpha_2 - 1) + b_0 a_2 = 0 \Rightarrow b_0 a_2^2 = 0 \]
\[ \Rightarrow \alpha_2 = 0 \quad \text{so} \quad b_0 \text{ arbitrary} \]

(\(\text{to } O(x^{-1})\)):
\[ b_1 (\alpha_2 + 1)a_2 + b_1 (\alpha_2 + 1) + 2a_1 = 0 \]
\[ \Rightarrow b_1 = -2a_1 = 0 \]

(\(\text{to } O(x^0)\)):
\[ b_2 (\alpha_2 + 2)(\alpha_2 + 1) + b_2 (\alpha_2 + 2) + b_0 + 4a_2 = 0 \]
\[ \Rightarrow 4b_2 + b_0 = -4a_2 \]
\[ \Rightarrow b_2 = \frac{-4a_2 - b_0}{4} \]
\[ \Rightarrow \alpha_2 = \frac{-b_0}{4} \]

This works & gives a unique solution.

**Case \(\alpha = 2\):**
\[ f_2(x) = \text{ln}|x| f_1(x) + x^2 \left(- a_2 - O(x^2) + \ldots \right) \]

**Case \(\alpha = 0\):**
\[ f_2(x) = \text{ln}|x| f_1(x) + b_0 - \frac{(a_2 + b_0)}{4} x^2 + \ldots \]

These solutions actually recover...
so finally, we get that the general solution will be a linear combination of \( f_1(x) \) and \( f_2(x) \) where
\[
\begin{align*}
  f_1(x) &= a_0 \left( 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots \right) & \text{the regular solution} f_1(x) \\
  f_2(x) &= b_0 \left( \ln|x| f_1(x) - a_2 x^2 + \cdots \right) & \text{the singular solution } f_2(x).
\end{align*}
\]

**Example 4:** Frobenius series for an irregular-singular pt?

We now study what happens when we try to expand around an irregular singular pt.

Consider \( \frac{d^2 f}{dx^2} + \frac{f}{x^4} = 0 \) near \( x = 0 \)

\[
\rightarrow \text{if we try } f(x) = \sum_{k=0}^{\infty} a_k x^{\alpha + k} \text{ then}
\]

\[
\sum_{k=0}^{\infty} a_k (\alpha + k)(\alpha + k - 1) x^{\alpha + k - 2} + \sum_{k=0}^{\infty} a_k x^{\alpha + k - 4} = 0
\]

\( \rightarrow \) the lowest term is \( O(x^{-4}) \) but it is not compensated by any term anywhere else \( \rightarrow \)

This would require \( a_0 = 0 \), but by definition we must have \( a_0 \neq 0 \) \( \rightarrow \) inconsistency

This shows that the Frobenius series don't work around an irregular singular pt.

However, if we remember that with \( s = \frac{1}{x} \), the equation becomes
\[
\frac{d^2 f}{ds^2} + \frac{2}{s} \frac{df}{ds} + f = 0,
\]

which happens to have a Frobenius series, \( \rightarrow \)

Eventually the solution \( f(s) = \frac{a_0}{s} \cos + \frac{b_0}{s} \sin \)

\( \rightarrow f(x) = a_0 x \cos(\frac{1}{x}) + b_0 x \sin(\frac{1}{x}) \)

It should be obvious from this that this function is not differentiable at \( x = 0 \) \( \rightarrow \) cannot have a series near there.