Example 3: An internal boundary layer.

Consider the equation $\frac{d^2f}{dx^2} + x \frac{df}{dx} + xf = 0$

$$f(-1) = e \quad f(0) = \frac{3}{2}$$

As before we can interpret this as the end-product of an advection/diffusion process, where the advection velocity is $v(x) = -x$.

$\begin{array}{c}
| \hline
| v(x) | \hline
| -1 | 1 | x \\
\end{array}$

$\Rightarrow$ we expect a boundary layer at $x=0$. However, since the actual boundaries are now at $x=1$ and $x=-1$, thus is an internal layer.

Let's look at the outer expansion. This time, "outer" means for $x$ out of the BL, that is, not too small.

Let $f = f_0 + ef_1 + \ldots$

$\Rightarrow$ To lowest order we have

$$x \frac{df_0}{dx} + xf_0 = 0 \Rightarrow f_0 = Ke^{-x}$$

Clearly, we cannot fit both BCs to this $f_0$ function. However, we can create two pieces of the outer expansion as

$$\begin{cases} f_0^+(x) = Ke^{-x} & \text{for } x > 0 \quad (x \text{ not in BL}) \\ f_0^-(x) = Ke^{-x} & \text{for } x < 0 \quad (x \text{ not in BL}) \end{cases}$$

$f_0^+$ then satisfies the BC at $x=1 \Rightarrow K^+ = 2$

$f_0^-$ then satisfies the BC at $x=-1 \Rightarrow K^- = 1$
The inner expansion is determined as usual. First assume that the inner vaneclde is \( s = \frac{x}{\varepsilon^2} \), then select \( p \) using Van Dyke's formula:

\[
\varepsilon \frac{d^2 f}{dx^2} + x \frac{df}{dx} + xf = 0 \Rightarrow \varepsilon^{1-2p} \frac{d^2 f}{ds^2} + s \frac{df}{ds} + \varepsilon^p s f = 0
\]

Clearly the 3rd term is much smaller than the second for \( p > 0 \) \( \Rightarrow \) we then want a balance between the first 2 second terms \( \Rightarrow \) take \( \varepsilon = 1 \) \( \Rightarrow \) \( p = \frac{1}{2} \)

The leading term of the inner expansion must then satisfy

\[
\frac{d^{2m}}{ds^2} + s \frac{df^m}{ds} = 0
\]

\( \Rightarrow \) let \( g = \frac{df^m}{ds} \) \( \Rightarrow \) \( \frac{dg}{ds} + sg = 0 \)

\( \Rightarrow \) \( \frac{dg}{g} = -sd \) \( \Rightarrow \) \( \ln g = -\frac{s^2}{2} + C \)

\( \Rightarrow \) \( g = k^m e^{-\frac{s^2}{2}} \)

\( \Rightarrow \) \( f^m_o = \int_0^s k^m e^{-\frac{s^2}{2}} ds' + f^m_o(0) \)

\( = \sqrt{\frac{\pi}{2}} k^m e f(\frac{s}{\sqrt{2}}) + f^m_o(0) \)

Prandtl's matching condition requires that

\( \lim_{s \to \infty} f^m_o(s) = \lim_{x \to 0^+} f^m_{outer}(x) = k^+ = 2 \)

and \( \lim_{s \to -\infty} f^m_o(s) = \lim_{x \to 0^-} f^m_{outer}(x) = k^- = 1 \)

Since \( \lim_{x \to +\infty} ef(x) = 1 \) and \( \lim_{x \to -\infty} ef(x) = -1 \),

This implies that

\[
\begin{align*}
\int k^m \sqrt{\frac{\pi}{2}} + f^m_o(0) &= 2 \\
-\int k^m \sqrt{\frac{\pi}{2}} + f^m_o(0) &= 1
\end{align*}
\]

\( \Rightarrow \) \( \int k^m \frac{\sqrt{\pi}}{2} + f^m_o(0) = \frac{3}{2} \)

\( \Rightarrow \) \( k^m = \frac{3}{2} \)
so finally, \( f_0^m(s) = \frac{3}{2} + \frac{1}{2} \text{erf}(\frac{s}{\sqrt{2}}) \)

Because we now have 3 components in the solution instead of just two, the formula for the composite expansion is a bit different but based on the same idea:

\[
\begin{align*}
    f_{\text{composite}}(x) &= f_+^m(x) + f_0^m(x) - \lim_{s \to \infty} f_+^m(s) \quad \text{if } x > 0 \\
    &= f_-^m(x) + f_0^m(x) - \lim_{s \to -\infty} f_-^m(s) \quad \text{if } x < 0 \\
    &= e^{-x} + \frac{3}{2} + \frac{1}{2} \text{erf}(\frac{x}{\sqrt{2}e}) - 2 = 2e^{-x} - \frac{1}{2} + \frac{1}{2} \text{erf}(\frac{x}{\sqrt{2}e})
\end{align*}
\]

which is not very pretty. However, we also see that

\[
    f_{\text{composite}}(x) = \left( \frac{3}{2} + \frac{1}{2} \text{erf}(\frac{x}{\sqrt{2}e}) \right) e^{-x}
\]

has all the right behavior. This is an equivalent, but more compact composite expansion.

---

**Example 4:** Boundary layers on each side:

This example illustrates that things are not necessarily so straightforward.

**Boundary:**
\[
    \begin{align*}
        \int e^{x''} - 2(x - \frac{1}{2}) e' &= 0 \\
        f(0) &= 0 \\
        f(1) &= b
    \end{align*}
\]

- The outer expansion is of the form \( f_0 + f_1 + \ldots \)
- So \(-2(x - \frac{1}{2}) f'_0 = 0 \Rightarrow f_0(x) = \text{constant}\)

However, we cannot match \( f_0 \) to the boundary conditions since each of them is within its respective boundary layer.

\( \Rightarrow \) Here we say that the outer expansion is "detached" from the boundaries.

\( \Rightarrow \) Let's hope that we can find the constant by matching with the two inner expansions.
Now let's deal with the inner expansion near $x = 0$. Call it $f_0^{(0)}$.

Let $s = \frac{x}{\varepsilon^{1/2}} \Rightarrow$

$$\varepsilon^{1-2p} \frac{d^2 f_0^{(0)}}{ds^2} + 2 \left( \varepsilon^{p} s - \frac{1}{2} \right) \varepsilon^{-p} \frac{df_0^{(0)}}{ds} = 0 \quad \Rightarrow \quad \varepsilon^p s \text{ is always } \ll 1$$

**Principle of least degeneracy:** we require that

$$\varepsilon^{1-2p} = \varepsilon^{-p} \Rightarrow \quad 1-2p = -p \Rightarrow \quad p = 1$$

Then, to lowest order

$$\frac{d^2 f_0^{(0)}}{ds^2} + \frac{df_0^{(0)}}{ds} = 0 \Rightarrow f_0^{(0)}(s) = A^{(0)} e^{-s} + B^{(0)}$$

To satisfy the BC $f_0^{(0)}(s=0)$ we take $A^{(0)} + B^{(0)} = 0$

$$f_0^{(0)}(s) = A^{(0)} (e^{-s} - 1) + B^{(0)}$$

Finally, let's deal with the inner expansion near $x = 1$;

Call it $f_0^{(1)}(\eta)$

Let $\eta = \frac{1-x}{\varepsilon^{1/2}} \Rightarrow \frac{d}{dx} = \frac{d\eta}{dx} \frac{d}{d\eta} = -\frac{1}{\varepsilon^{1/2}} \frac{d}{d\eta}$

$$\Rightarrow \quad \varepsilon^{1-2p} \frac{d^2 f_0^{(1)}}{d\eta^2} + 2 \left( 1 - \varepsilon^{p} \eta - \frac{1}{2} \right) \varepsilon^{-p} \frac{df_0^{(1)}}{d\eta} = 0$$

$$\Rightarrow \quad \text{For the same reasons as above, we need}$$

$$\varepsilon^{1-2p} = \varepsilon^{-p} \Rightarrow p = 1 \text{ so } \text{ to lowest order}$$

$$\frac{d^2 f_0^{(1)}}{d\eta^2} - \frac{df_0^{(1)}}{d\eta} = 0 \Rightarrow f_0^{(1)}(\eta) = A^{(1)} e^{\eta} + B^{(1)}$$

To satisfy the BCs, we need

$$f_0^{(1)}(0) = A^{(1)} + B^{(1)} = 0 \Rightarrow B^{(1)} = -A^{(1)}$$

So

$$f_0^{(1)}(\eta) = A^{(1)} (e^{\eta} - 1) + b$$

(recall that $\eta = 0$ when $x = 1$)
So far, everything looks fine and it seems we just have to match them all to one another.

However, even without doing the calculation we now see that there is a problem: we will be getting 2 matching conditions, but we have 3 variables to fit! (one from the outer expansion, and one for each inner expansion).

→ This problem has an \( \infty \) of possible solutions!!

The issue with this problem is that it is almost ill-posed. This is not a problem with the expansions, but rather, a problem with the original equation.

Not all problems with 2 boundary layers suffer from this issue, however. Let’s look at a final example that is well-posed.

**Example 5**

\[
\begin{aligned}
& ef” - f = -2\sin(x-\frac{1}{2}) \\
& f(0) = 0 \quad f(1) = 0.
\end{aligned}
\]

In this example, the “steady-state of a time-dependent advection/diffusion problem” analogy does not help us find the location of the PE, since there is no \( \frac{df}{dx} \) term in the equation.

However, note that when \( \varepsilon = 0 \) \( f = 2\sin(x-\frac{1}{2}) \) which does not satisfy either of the boundary conditions \( \Rightarrow \) the system will therefore have to have one boundary layer on each side to fit the outer expansion to the B.C.s.
Outer expansion: if 

we have, to lowest order \( f_0(x) = 2 \sin(x - \frac{1}{2}) \)

→ this time, no arbitrary constants!

Inner expansion near \( x = 0 \):

\[ \varepsilon^{1-2p} \frac{d^2 f_0^{(0)}}{ds^2} - f_0^{(0)} = -2 \sin(\varepsilon^{p} s - \frac{1}{2}) \] \( \varepsilon^{p} s \ll \frac{1}{2} \) always

→ Van Dyke's matching condition requires \( \varepsilon^{1-2p} = \varepsilon^0 \)

\[ 1 - 2p = 0 \implies p = \frac{1}{2} \]

To lowest order, we then get

\[ \frac{d^2 f_0^{(0)}}{ds^2} - f_0^{(0)} = 2 \sin(\frac{1}{2}) \]

So \( f_0^{(0)}(s) = A^{(0)} e^s + B^{(0)} e^{-s} - 2 \sin(\frac{1}{2}) \)

To fit the BC at \( s = 0 \) (\( x = 0 \)) we have

\[ A^{(0)} + B^{(0)} - 2 \sin(\frac{1}{2}) = 0 \implies B^{(0)} = -A^{(0)} - 2 \sin(\frac{1}{2}) \]

\[ f_0^{(0)}(s) = A^{(0)} (e^s - e^{-s}) - 2 \sin(\frac{1}{2})(1 - e^{-s}) \]

Inner expansion near \( x = 1 \)

\[ \eta = \frac{1-x}{\varepsilon^p} \]

\[ \varepsilon^{1-2p} \frac{d^2 f_0^{(1)}}{d\eta^2} - f_0^{(1)} = -2 \sin(1 - \eta e^p - \frac{1}{2}) \]

Again, thus requires \( p = \frac{1}{2} \) → to lowest order

\[ \frac{d^2 f_0^{(1)}}{d\eta^2} - f_0^{(1)} = -2 \sin(\frac{1}{2}) \]
The solution is \( f^{(1)}_0(\eta) = A^{(1)} e^\eta + B^{(1)} e^{-\eta} + 2\sin(\frac{x}{2}) \).

At \( x=1 \) \((\eta=0)\), \( f^{(1)}_0(0) = 0 \Rightarrow A^{(1)} + B^{(1)} + 2\sin(\frac{1}{2}) = 0 \)
so \( B^{(1)} = -A^{(1)} - 2\sin(\frac{1}{2}) \Rightarrow \)
\[
 f^{(1)}_0(\eta) = A^{(1)} (e^\eta - e^{-\eta}) + 2\sin(\frac{1}{2})(1 - e^{-\eta})
\]

This time, the two matching conditions \( x=0 \) and \( x=1 \) will indeed constrain the two unknown constants \( A^{(0)} \) and \( A^{(1)} \). Let's proceed with Prandtl's matching condition near \( x=0 \) first. We want

\[
 \lim_{x \to 0} f^{\text{outer}}(x) = \lim_{\eta \to \infty} f^{(0)}(\eta)
\]

\[
 = \lim_{x \to 0} 2\sin(\frac{x}{2}) = \lim_{\eta \to \infty} A^{(0)} (e^\eta - e^{-\eta}) - 2\sin(\frac{1}{2})(1 - e^{-\eta})
\]

\[
 -2\sin(\frac{1}{2}) = \lim_{\eta \to \infty} A^{(0)} e^\eta - 2\sin(\frac{1}{2})
\]

The only way this can work is if \( A^{(0)} = 0 \)

\[
 f^{(0)}(\eta) = -2\sin(\frac{1}{2})(1 - e^{-\eta}) \Rightarrow f^{(0)}_0(x) = -2\sin(\frac{1}{2})(1 - e^{-\frac{x}{2}})
\]

Near \( x=1 \), we want

\[
 \lim_{x \to 1} f^{\text{outer}}(x) = \lim_{\eta \to \infty} f^{(1)}(\eta)
\]

\[
 = \lim_{x \to 1} 2\sin(\frac{x}{2}) = \lim_{\eta \to \infty} A^{(1)} (e^\eta - e^{-\eta}) + 2\sin(\frac{1}{2})(1 - e^{-\eta})
\]

\[
 2\sin(\frac{1}{2}) = \lim_{\eta \to \infty} A^{(1)} e^\eta + 2\sin(\frac{1}{2})
\]

\[
 = \text{again this will only work if } A^{(1)} = 0
\]

\[
 \Rightarrow f^{(1)}_0(\eta) = 2\sin(\frac{1}{2})(1 - e^{-\eta}) \Rightarrow f^{(1)}_0(x) = 2\sin(\frac{1}{2})(1 - e^{-\frac{x}{2}})
\]
We see that in this case, there is a solution even though the outer expansion is detached from the boundaries. That's because the outer expansion itself has no arbitrary constants.

Again, because we have 3 parts to match, the composite expansion formula is a little different.

\[
\text{Inner} (x) = \text{outer} (x) + \lim_{s \to \infty} f^{(s)}(x) = \lim_{s \to \infty} f^{(s)}(\xi) + \lim_{s \to \infty} f^{(s)}(\eta)
\]

\[
= \alpha \sin \left( x - \frac{1}{2} \right) + \alpha \sin \left( \frac{1}{2} \right) \left( e^{-\frac{X}{2l^2}} - 1 \right) + \alpha \sin \left( \frac{1}{2} \right) \left( 1 - e^{-\frac{X}{2l^2}} \right) - \alpha \sin \left( \frac{1}{2} \right)
\]

\[
= \alpha \sin \left( x - \frac{1}{2} \right) + \alpha \sin \left( \frac{1}{2} \right) \left( e^{-\frac{X}{2l^2}} - e^{-\frac{X}{2l^2}} \right)
\]

**Higher-order matching: Van Dyke's matching principle**

So far we have only been concerned with obtaining 0th order composite expansions (1-term expansion). We saw that Prandtl's matching condition does not work "as is" if we want to keep more terms, however.

A nice technique that works most of the time for higher-order matched asymptotic expansions is Van-Dyke's matching principle.

Idea:  
- **Taylor expand** the outer expansion as \( x \to x_{BL} \) (where \( x_{BL} \) is position of BL)
- Substitute \( x \) with \( x_{BL} + se^p \)

\[
\left( s = \frac{x - x_{BL}}{e^p} \rightarrow x = x_{BL} + se^p \right)
\]
This procedure gives the "inner limit of the outer expansion".

- Then compare this with the expansion of the inner function as \( s \to \pm \infty \) (depending on the behavior of the BL)
- This gives "the outer limit of the inner expansion".

- Match term by term, and voila!

**Example 1:** Let's go back to

\[
\begin{align*}
\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y &= 0 \\
y(0) &= 0 \\
y(1) &= 1
\end{align*}
\]

We had:

\[
\begin{align*}
y_{\text{outer}}(x) &= e^{-x} + e^{(1-x)}e^{-x} \quad (2 \text{term}) \\
y_{\text{inner}}(s) &= A_0 (1 - e^{-s}) + eA_1 (e^{-s} - 1) \\
&\quad - eA_0 s (1 + e^{-s}) \quad (2 \text{term})
\end{align*}
\]

1. **Taylor-expand** \( y_{\text{outer}}(x) \) as \( x \to 0 \):

\[
y_{\text{outer}}(x) = e^{-x} + e^{(1-x)}e^{-x} = e^{-x} + e^{-x} \frac{x^2}{2!} + \ldots
\]

2. **Substitute** \( x = es \):

\[
y_{\text{outer}}(s) = e^{-es} + e^{(1-es)}e^{-es} \frac{e^2 s^2}{2} + \ldots
\]

\[
= e^{-es} + e^{-es} \left[-es + e^{es} \right] + O(e^2)
\]

3. **Take the taylor expansion of** \( y_{\text{inner}}(s) \) as \( s \to \infty \)

Note that as \( s \to \infty \), all terms in \( e^{-s} \) are asymptotically small so we simply have

\[
y_{\text{inner}}(s) = A_0 + eA_1 (-1) - eA_0 s + O(e^2)
\]

\[
= A_0 + e \left[-A_1 - A_0 s \right]
\]
4. Match order by order.
\[
\begin{align*}
A_0 &= e \\ -A_1 - A_0 s &= e(1-s)
\end{align*}
\]
\[
\Rightarrow \begin{cases} A_0 = e \\ A_1 = -es - e(1-s) = -e \end{cases}
\]

Note how the terms in $s$ cancel out. This is crucial, but not always guaranteed.
If this does not happen, then either we made an error in the algebra (always a possibility) or it is a symptom of the failure of Van Dyke's matching method.

See [hunch textbooks for examples.](#)

Finally, we can construct the composite expansion as
\[
y(x) = \lim_{\text{2-tem}} \left( y_{\text{outer}}(x) + y_{\text{inner}}(x) \right) - \lim_{\text{2-tem}} \left( \frac{\text{their common limit}}{2-\text{tem}} \right).
\]
\[
= e^{1-x} + e(1-x)e^{1-x} + e(1+\varepsilon)(1-e^{-\frac{x}{\varepsilon}})
\]
\[
-\varepsilon e^{\frac{x}{\varepsilon}}(1+e^{-\frac{x}{\varepsilon}}) - e - \varepsilon e(1-x)
\]
\[
= e^{1-x} - e^{1-x} - \varepsilon e^{\frac{x}{\varepsilon}}
\]
\[
+ \varepsilon \left[ (1-x)e^{1-x} - e^{1-x} \right] + \ldots
\]

Note the appearance of an extra term in the other-order composite expansion — latter was not uniform.
Example 2. A nonlinear problem: same procedure...

Let \[ \begin{align*}
\epsilon y'' + y' + y^2 &= 0 & \epsilon < 0 \\
y(0) &= 1, & y(1) &= 0
\end{align*} \]

First let's use the new small parameter \( \eta = -\epsilon \) then

\[ \begin{align*}
\eta y'' - y' - y^2 &= 0 & \eta > 0 \\
y(0) &= 1, & y(1) &= 0
\end{align*} \]

This then shows that the advection velocity is \( >0 \), so the boundary layer is to the right (at \( x=1 \)).

Outer expansion:

Let \( y = y_0 + \eta y_1 + \cdots \)

\[ \Rightarrow \text{0th: } \begin{align*}
y_1' + y_0^2 &= 0 \\
y_0(0) &= 1
\end{align*} \]

\( \Rightarrow \frac{dy_0}{y_0^2} = dx \Rightarrow \frac{1}{y_0} = x + k \Rightarrow y_0 = \frac{1}{k+x} \]

To apply the BC we need \( 1 = \frac{1}{k} \) so \( y_0 = \frac{1}{1+x} \)

\[ \Rightarrow \text{let's: } \begin{align*}
y_1'' - y_1' - 2y_0y_1 &= 0 \\
y_1(0) &= 0
\end{align*} \]

\( \Rightarrow y_1' + 2 \frac{y_1}{1+x} = \left[ -\frac{1}{(1+x)^2} \right]' = \frac{2}{(1+x)^3} \]

Let's use an integrating factor:

\( \mu(x) = e^{\int \frac{2}{1+x} \, dx} = e^{2 ln(1+x)} = (1+x)^2 \)

\( \Rightarrow \frac{d}{dx} \left( (1+x)^2 y_1 \right) = \frac{2}{1+x} \)
\[(1+x)^2 y_1 = 2 \ln(1+x) + \kappa\]

to apply the BC, \(y_1(0) = 0 \Rightarrow \kappa = 0\) so

\[y_1(x) = \frac{2 \ln(1+x)}{(1+x)^2}\]

So the outer expansion, to this order, is

\[y_{\text{outer}}(x) = \frac{1}{1+x} + 2 \eta \frac{\ln(1+x)}{(1+x)^2} + o(\eta^2)\]

- Inner expansion. The BL is at \(x = 1\) so we let

\[s = \frac{x-1}{\eta^p} \Rightarrow \text{let's use Van Dyke's principle to find } p = \eta^{-2p} d^2 y \frac{ds^2}{ds^2} - \eta^{-p} d y \frac{ds}{ds} - y^2 = 0\]

For \(p > 0\), (and \(y\) of order \(1\)), \(\eta^{-p}\) is always much larger than 1. So, to the lowest order, we should have a balance between

\[\eta^{-2p} = \eta^{-p} \Rightarrow 1 - 2p = -p \Rightarrow 1 - p = 0 \Rightarrow p = 1\]

So let \(s = \frac{x-1}{\eta}\)

Also let \(y = y_0^\text{in} + \eta y_1^\text{in} + \ldots\)

To lowest order:

\[\frac{1}{E} \frac{d^2 y_0^\text{in}}{ds^2} - \frac{1}{E} \frac{dy_0^\text{in}}{ds} = 0, \text{ with } y_0^\text{in}(0) = 0\]

\[\Rightarrow y_0^\text{in}(s) = A_0 e^s + B_0\]

to match this to the boundary condition at \(x = 1\) (\(s = 0\)) we need \(A_0 + B_0 = 0 \Rightarrow y_0^\text{in}(s) = A_0 (e^s - 1)\)
To the next order:

\[ \frac{d^2 y_1^{(m)}}{d s^2} - \frac{d y_1^{(m)}}{d s} - (y_0^{(m)})^2 = 0 \quad \text{with} \ y_1^{(m)}(0) = 0 \]

\[ \Rightarrow \quad \frac{d^2 y_1^{(m)}}{d s^2} - \frac{d y_1^{(m)}}{d s} = A_0^2 (e^s - 1)^2 = A_0^2 (e^{2s} - 2e^s + 1) \]

The general solution to the homogeneous problem is

\[ y_1^{(m)}(s) = A_1 e^s + B_1 \]

The particular solution will be of the form

\[ y_{ps} = k_1 e^s + k_2 e^{2s} + k_3 e^{3s} + k_4 \]

\[ \Rightarrow \quad \frac{d y_{ps}}{d s} = k_1 e^s + 2k_2 e^{2s} + 3k_3 e^{3s} \]

\[ \frac{d^2 y_{ps}}{d s^2} = 2k_2 e^s + 4k_3 e^{2s} + 6k_4 e^{3s} \]

\[ \Rightarrow \quad 2k_2 = A_0^2, \quad k_3 = -2A_0, \quad k_1 = A_0^2 \]

and so finally,

\[ y_1^{(m)}(s) = A_1 e^s + B_1 + A_0^2 \left( -s - 2se^s + \frac{1}{2} e^{2s} \right) \]

To fit the BC, we need:

\[ A_1 + B_1 + \frac{1}{2} A_0^2 = 0 \]

so

\[ B_1 = -\frac{1}{2} A_0^2 - A_1 \]

\[ \Rightarrow \quad y_1^{(m)}(s) = A_1 (e^s - 1) + A_0^2 \left( -s - 2se^s + \frac{1}{2} e^{2s} - \frac{1}{2} \right) \]

\[ = A_1 (e^s - 1) + A_0^2 \left( \frac{1}{a} (e^{2s} - 1) - s(1 + e^s) \right) \]

And finally:

\[ y_1^{(m)}(s) = A_1 (e^s - 1) + A_0^2 \left( \frac{1}{a} (e^{2s} - 1) - s(1 + e^s) \right) \]
Let's now take the Taylor expansion of \( y_{\text{inner}}(s) \) as \( s \to -\infty \)

\[
\lim_{s \to -\infty} y_{\text{inner}}(s) = -A_0 - \eta A_1 + \eta A_0^2 \left( -\frac{1}{2} - s \right) + o(\eta^2)
\]

And the Taylor expansion of \( y_{\text{outer}} \) as \( x \to 1 \):

\[
y_{\text{outer}}(x) = y_{\text{outer}}(1) + (x-1) \frac{dy_{\text{outer}}}{dx} \bigg|_{x=1}
\]

\[
= \frac{1}{2} + \eta \ln^2 \left( \frac{a}{2} \right) + (x-1) \left[ -\frac{1}{2} + \frac{2 \eta}{(2)^3} + 2 \ln^2 \left( \frac{a}{2} \right) \right] + o(\eta^2)
\]

Matching the two expansions, using \( x - 1 = \eta s \), yields:

\[
\lim_{s \to -\infty} y_{\text{outer}}(s) = \frac{1}{2} + \frac{\ln^2 \eta}{a} \eta - \frac{\eta s^2}{4} + o(\eta^2)
\]

\[
\lim_{s \to -\infty} y_{\text{inner}}(s) = -A_0 - \eta A_1 + \eta A_0^2 \left( -\frac{1}{2} - s \right) + o(\eta^2)
\]

At order \( \eta^0 \), we get

\[
A_0 = \frac{-1}{2}
\]

At order \( \eta^1 \), we get

\[
\frac{\ln^2 \eta}{a} - \frac{\eta s^2}{4} = -A_1 - \left( \frac{1}{2} + s \right) A_0^2
\]

\[
= -A_1 - \left( \frac{1}{2} + s \right) \frac{1}{4}
\]

\[
\Rightarrow \quad \frac{\ln^2 \eta}{a} = -A_1 - \frac{1}{8} \Rightarrow \quad A_1 = -\frac{\ln^2 \eta}{2} - \frac{1}{8}
\]

So

\[
y_{\text{inner}}(s) = -\frac{1}{a} (e^s - 1) - \eta \left( \frac{\ln^2 \eta}{2} + \frac{1}{8} \right) (e^s - 1)
\]

\[
+ \frac{\eta}{4} \left[ \frac{1}{2} (e^{2s} - 1) - s (1 + e^s) \right]
\]

And finally, the composite 2-term expansion is:

\[
y_{\text{composite}}(x) = \frac{1}{1 + x} + 2 \eta \frac{\ln(1 + x)}{(1 + x)^2} - \frac{1}{a} (e^{x+1} - 1) - \eta \left( \frac{\ln^2 \eta}{2} + \frac{1}{8} \right) (e^x - 1)
\]
\[
\frac{\eta}{4} \left[ \frac{1}{2} \left( e^{\frac{a(x-1)}{2}} - 1 \right) - \frac{x-1}{\eta} \left( 1 + 2e^{\frac{-x}{\eta}} \right) \right] + O(\eta^2)
\]
\[
- \left( \frac{1}{2} + \eta \left( \frac{\ln 2}{2} - \frac{\eta}{4} \frac{x-1}{\eta} \right) \right) - 2 \text{-term common limit of the two expansions.}
\]
\[
= \frac{1}{1+x} - \frac{1}{a} e^{\frac{x-1}{2}} - \frac{x-1}{2} e^{\frac{x-1}{2}}
\]
\[
+ \eta \left[ \frac{2 \ln(1+x)}{2} - \left( \frac{\ln 2}{2} + \frac{1}{8} \right) e^{\frac{x-1}{2}} - \frac{1}{8} \left( e^{\frac{x-1}{2}} - e^{\frac{1}{2}} \right) \right]
\]
\[
= \frac{1}{1+x} - \frac{x}{2} e^{\frac{x-1}{2}}
\]
\[
+ \eta \left[ \frac{2 \ln(1+x)}{2} - \frac{\ln 2}{2} e^{\frac{x-1}{2}} + \frac{1}{8} \left( e^{\frac{x-1}{2}} - e^{\frac{1}{2}} \right) \right]
\]