Problem 1: Pick either Problem 1A or Problem 1B

Problem 1A: There are 3 possible roots of this equation:

\[ \varepsilon x^3 + 4x^2 + 2x + \varepsilon = 0 \]

Using any method of your choice, find a 2-term expansion for each of them.

Problem 1B: Using any method of your choice, find the period of the following oscillator to second-order in \( \varepsilon \) (i.e., including the correction term in \( \varepsilon^2 \))

\[ \frac{d^2 \theta}{dt^2} = -\theta + \varepsilon \theta^3 \]

\[ f(0) = 1 \]
\[ f'(0) = 0 \]

You may need to use the following identities:

\[ \cos^3(x) = \frac{1}{4} (3\cos x + \cos 3x) \]
\[ \sin^3(x) = \frac{1}{4} (3\sin x - \sin 3x) \]
\[ \cos^2(x) = \frac{1}{2} (1 + \cos 2x) \]
\[ \sin^2(x) = \frac{1}{2} (1 - \cos 2x) \]
\[ \cos a \cos b = \frac{1}{2} \left[ \cos (a-b) + \cos (a+b) \right] \]
\[ \cos a \sin b = \frac{1}{2} \left[ \sin (a+b) - \sin (a-b) \right] \]
\[ \sin a \sin b = \frac{1}{2} \left[ \cos (a-b) - \cos (a+b) \right] \]
Problem 2. Using the method of multiple scales, with fast time $T_0 = t$ and slow time $T_1 = e^{2t}$ (note the $e^2$!), show that the solution to

$$\frac{d^2f}{dt^2} + \left(1 + e^2 + e^{2t}\right) f = 0$$

has the form:

$$f(t) = A(T_1) \cos t + B(T_1) \sin t + O(e)$$

where

$$\int \frac{dB}{dT_1} = \frac{1}{2} \left(\frac{5}{12} - k\right) A$$

$$\int \frac{dA}{dT_1} = \frac{1}{2} \left(\frac{1}{12} + k\right) B$$

Note: You do not have to solve these equations for $A$ and $B$.

1. Assume $f = f_0(T_0, T_1) + e f_1(T_0, T_1) + e^2 f_2(T_0, T_1)$

Hunt: The $O(e)$ equation does not lead to any secular term. These are given by the $O(e^2)$ equation.

Problem 3. Obtain the 1-term composite expansion for the boundary layer problem

$$\begin{cases}
\varepsilon \frac{d^2f}{dx^2} + \frac{df}{dx} - 2x e^{-f} = 0 \\
f(0) = 0 \\
f(1) = 1
\end{cases}$$
Problem 4  Obtain an approximation for the large eigenvalues of

\[ \begin{align*}
\frac{d^2 f}{dx^2} + (\lambda^2 x + \lambda) f &= 0 \\
\int_0^{\lambda_1} f(x) dx &= 0 \\
\int_0^{\lambda_2} f(x) dx &= 0
\end{align*} \]

Problem 5  Using the fact that \( \int_0^\infty \cos u^2 du = \frac{1}{2} \sqrt{\frac{\pi}{2}} \),

show that

\[ \int_0^\infty \frac{1}{2 + t} \cos (\lambda (4t - t^4)) dt = \frac{1}{3} \sqrt{\frac{\pi}{12\lambda}} \left( \cos 3\lambda + \sin 3\lambda \right) + \text{h.o.t.} \]