Required Problems

1. Consider using Jeffreys prior for the unknown mean \( \lambda \) for Poisson data (e.g., the cookies example).
   
   (a) First, note that the likelihood for a single observation is \( f(y|\lambda) = \frac{1}{y!} \lambda^y e^{-\lambda} \). Take the natural log of this to find the log-likelihood. Now take the second derivative with respect to \( \lambda \).
   
   (b) Find the expected value of the second derivative of the log-likelihood (in terms of \( \lambda \)),
   
   \[ E \left[ \frac{\partial^2}{\partial \lambda^2} \log f(y|\lambda) \right], \]
   
   by using the fact that \( Y \sim \text{Pois}(\lambda) \).
   
   (c) Find the Jeffreys prior, which is proportional to the square root of the negative of the expected value of the second derivative of the log-likelihood, i.e.,
   
   \[ f(\lambda) \propto \sqrt{-E \left[ \frac{\partial^2}{\partial \lambda^2} \log f(y|\lambda) \right]} \]
   
   (d) What limiting Gamma prior would this be equivalent to?
   
   (e) Find the posterior distribution (in closed form) using this prior, with data \( y_1, \ldots, y_n \).

2. Recall that the density for the Pareto distribution is
   
   \[ f(x|\alpha, \beta) = \alpha \beta^\alpha x^{-(\alpha+1)} I_{\{x \geq \beta\}} \]
   
   (a) Show that the inverse CDF for the Pareto is \( F^{-1}(p) = \beta(1-p)^{-1/\alpha} \).
   
   (b) Find the mean and variance of a Pareto distribution with parameters \( \alpha = 4 \) and \( \beta = 6 \) by Monte Carlo estimation (generate samples using the inverse CDF method), and compare them to the theoretical values of \( E[X] = \frac{\alpha \beta}{\alpha-1} \) and \( \text{Var}(X) = \frac{\alpha \beta^2}{(\alpha-1)^2(\alpha-2)} \).
   
   (c) Define a new distribution by \( Y = \log(X - \beta) \) where \( X \) has a Pareto(\( \alpha, \beta \)) distribution (and log is the natural log). Estimate the mean and variance of this distribution for \( \alpha = 4 \) and \( \beta = 6 \) (transform your sample from the previous part).

3. Use rejection sampling to draw 1000 samples from a Beta(3,5) distribution. Compute Monte Carlo estimates of the mean, median, and variance of the distribution and compare them to the theoretical values (or to the value from R for the median).

Optional Problem

4. Show that for a normal likelihood with unknown mean and unknown variance, the marginal posterior distribution for the mean is a scaled \( t \). You will probably want to use the form for the joint posterior that has the square already completed for \( \mu \). Note that the density function for a scaled \( t_\nu(\mu, \sigma^2) \) is
   
   \[ f(x|\nu, \mu, \sigma^2) = \frac{\Gamma \left[ \frac{1}{2}(\nu + 1) \right]}{\Gamma \left( \frac{\nu}{2} \right) \sqrt{\nu \pi \sigma^2}} \left[ 1 + \frac{1}{\nu \sigma^2} (x - \mu)^2 \right]^{-(\nu+1)/2} \]