Required Problems

1. One of the things that most bothers me about the Windows operating system is when it crashes when I’m trying to shut it down. It seems to be more likely to crash when I have put the computer on standby (sleep) earlier in the day. Suppose the probability that it crashes when it has been on standby earlier is 0.1, and the probability it crashes when it has not been on standby is 0.01, and that the probability that I put it on standby during a day’s usage is 0.4. If you just walked into the room in time to see me trying to shutdown and have the computer crash, what is the probability that I had put it on standby earlier that day?

2. Suppose you have an acquaintance that doesn’t understand the laws of probability, and they think that the probability it will rain tomorrow is 0.3, the probability it is overcast but doesn’t rain is 0.3, and the probability it is sunny without rain is 0.5, and that these are the only events under consideration. Suppose you make them the pair of bets (i) if it rains or is overcast you pay them $4, otherwise they pay you $6; (ii) if it is sunny you pay them $5, otherwise they pay you $5. Show that they should consider each of these to be fair bets based on their probabilities, and also show that you have made “Dutch book”, i.e., you are guaranteed to win money regardless of tomorrow’s weather.

3. Consider the case of flipping a coin 100 times and getting 56 heads. Denote by \( \theta \) the probability that the coin shows heads.

(a) Taking a frequentist approach, is it reasonable that this is a fair coin? The standard hypothesis testing approach declares an event to be “unusual” if the probability of getting a result as or more extreme is less than .05. If the event is declared unusual, then it would not be reasonable that the coin is fair. (For those of you familiar with hypothesis testing, technically we should be doing a two-sided test, but let’s keep things simple here by just looking at one tail).

i. Use R to find the exact probability of getting at least 56 heads on 100 flips of a fair coin. [Just show the R command and the result.] Is it reasonable that the coin is fair (i.e., is this probability at least .05)?

ii. Use the normal approximation to the binomial (via the central limit theorem) (also use the continuity correction if you are familiar with it) to compute the approximate probability of getting at least 56 heads on 100 flips of a fair coin. [Show your work in computing the normal approximation, then use R instead of looking up some number in a normal table.]

(b) Now take a Bayesian approach. Suppose you have a uniform prior for \( \theta \).

i. Compute your posterior distribution for \( \theta \).

ii. What is the posterior probability that \( \theta > .5 \)? [use R, since you should have a Beta distribution]

iii. Compute your posterior predictive distribution for a new coin flip [the easy way to do this is to compute \( P(Y = 1|X = 56) \), where \( Y \) is the new flip, and then note that \( P(Y = 0|X = 56) = 1 - P(Y = 1|X = 56) \)].

(c) What do you conclude about this coin?
Optional Problems

4. Consider three prisoners A, B, and C, exactly one of whom will be pardoned; the other two will be executed. Let $A$, $B$, and $C$, respectively denote the events that prisoner A, B, or C will be pardoned. In the absence of any other evidence to the contrary, it is reasonable to assume the “prior probabilities” $p(A) = p(B) = p(C) = \frac{1}{3}$. The warden now enters prisoner A’s cell and tells him that B will be executed. Assume that the warden will always truthfully tell A which one of B or C is being executed (if the other one is being pardoned), and that if both are being executed, he will pick one with equal probability to tell A.

(a) Are these assumptions reasonable?

(b) Use Bayes’ Theorem to show that the warden did not provide A with any useful information, i.e., the probability of A being pardoned conditioned on the information is unchanged.

(c) Prisoner C is listening through a hole in the wall. What is his updated probability of being pardoned?