Required Problems

1. Consider using Jeffreys prior for the unknown mean $\lambda$ for Poisson data (e.g., the cookies example).

   (a) First, note that the likelihood for a single observation is $f(y|\lambda) = \frac{1}{y!} \lambda^y e^{-\lambda}$. Take the natural log of this to find the log-likelihood. Now take the second derivative with respect to $\lambda$.

   (b) Find the expected value of the second derivative of the log-likelihood (in terms of $\lambda$), $E \left[ \frac{\partial^2}{\partial \lambda^2} \log f(y|\lambda) \right]$, by using the fact that $Y \sim \text{Pois}(\lambda)$.

   (c) Find the Jeffreys prior, which is proportional to the square root of the negative of the expected value of the second derivative of the log-likelihood, i.e.,

   $$f(\lambda) \propto \sqrt{-E \left[ \frac{\partial^2}{\partial \lambda^2} \log f(y|\lambda) \right]}.$$

   (d) What limiting Gamma prior would this be equivalent to?

   (e) Find the posterior distribution (in closed form) using this prior, with data $y_1, \ldots, y_n$.

2. Recall that the density for the Pareto distribution is

   $$f(x|\alpha, \beta) = \alpha \beta^\alpha x^{-(\alpha + 1)} I_{\{x \geq \beta\}}.$$

   (a) Show that the inverse CDF for the Pareto is $F^{-1}(p) = \beta (1 - p)^{-1/\alpha}$.

   (b) Find the mean and variance of a Pareto distribution with parameters $\alpha = 4$ and $\beta = 6$ by Monte Carlo estimation (generate samples using the inverse CDF method), and compare them to the theoretical values of $E[X] = \frac{\alpha \beta}{\alpha - 1}$ and $Var(X) = \frac{\alpha \beta^2}{(\alpha - 1)^2 (\alpha - 2)}$.

   (c) Define a new distribution by $Y = \log(X - \beta)$ where $X$ has a Pareto($\alpha, \beta$) distribution (and log is the natural log). Estimate the mean and variance of this distribution for $\alpha = 4$ and $\beta = 6$ (transform your sample from the previous part).

3. Use rejection sampling to draw 1000 samples from a Beta(2,3) distribution. Compute Monte Carlo estimates of the mean, median, and variance of the distribution and compare them to the theoretical values (or to the value from R for the median).

Optional Problem

4. Show that for a normal likelihood with unknown mean and unknown variance, the marginal posterior distribution for the mean is a scaled $t$. You will probably want to use the form for the joint posterior that has the square already completed for $\mu$. Note that the density function for a scaled $t_{\nu}(\mu, \sigma^2)$ is

   $$f(x|\nu, \mu, \sigma^2) = \frac{\Gamma \left[ \frac{1}{2}(\nu + 1) \right]}{\Gamma \left( \frac{\nu}{2} \right) \sqrt{\nu \pi \sigma^2}} \left[ 1 + \frac{1}{\nu \sigma^2} (x - \mu)^2 \right]^{-(\nu+1)/2}$$