Review of Random Variables (r.v.) {values} {prob.}

1. One at a time: real-valued r.v. (rv)

Any rv is uniquely characterized by its cumulative distribution function (CDF)

$F_Y(y) = P(Y \leq y)$. Nicely-behaved rv are either discrete or continuous.

(F is non-decreasing and $F_X(\leq 1)$. A discrete rv is equivalently characterized by its probability mass function (PMF) $P(Y = y)$.

Ex. $Y \sim$ Bernoulli $(\theta)$ parameter

$P(Y = y) = \left\{ \begin{array}{ll} \theta & \text{for } y = 1 \\ 1-\theta & \text{else} \end{array} \right\} = \theta^y(1-\theta)^{1-y}$

1.0

0

0 1
For a continuous RV $X$ with CDF $F_X(y)$, if you can find a function $f_X(y)$ such that $F_X(y) = \int_{-\infty}^{y} f_X(t) dt$

then $f_X(y)$ is (a) the density function of $X$; then $f_X(y) = F_X'(y)$. Immediately $F_X(y) = P(X \leq y) = \int_{-\infty}^{y} f_X(t) dt$, so probability for cont. RV is represented by area under the density function.

**Ex.** $X \sim \text{Uniform} (a, b)$ ($a < b$)

$$f_X(y) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq y \leq b \\ 0 & \text{else} \end{cases}$$
\[ F_Z(y) = \begin{cases} 0 & \text{for } y \leq a \\ \frac{y-a}{b-a} & a < y \leq b \\ 1 & \text{for } y \geq b \end{cases} \]

Example: \( Z \sim \text{Gaussian}(\mu, \sigma^2) = N(\mu, \sigma^2) \)

pdf

\[ f_Z(y) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} \]

cdf

\[ F_Z(y) = \left( \begin{array}{l} \text{closed form} \end{array} \right) \]

In this case, densities will typically not be denoted by \( f_Z(y) \) but by \( p_Z(y) \), or even more often by \( p(y) \) (infer w.r.t. from argument) e.g. \( p(\theta) \cdots p(\tau) \)

\( \circ \) 2 at a time (22 by extension)
The joint CDF of $X_1$ and $X_2$ is

$$F_{X_1, X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$$

If $X_1, X_2$ both discrete, make sense to talk about joint probability

$$p_{X_1, X_2}(x_1, x_2) = P(X_1 = x_1, X_2 = x_2)$$

If continuous, natural to consider joint density $f_{X_1, X_2}$ defined by

$$F_{X_1, X_2}(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X_1, X_2}(t_1, t_2) dt_2 dt_1$$

The marginal CDFs are given by

$$F_{X_1}(x_1) = \lim_{x_2 \to \infty} F_{X_1, X_2}(x_1, x_2)$$

& marginal density is given by
\[ f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1, Y_2}(y_1, y_2) \, dy_2 \]

(marginalizing over \( Y_2 \)).

Conditional density of \( Y_2 \) given \( Y_1 \) is

\[ f_{Y_2|Y_1}(y_2 | y_1) = \begin{cases} \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_1}(y_1)} & \text{if } f_{Y_1}(y_1) > 0 \\ 0 & \text{else} \end{cases} \]

From this it follows that if \( Y_1, Y_2 \) independent,

\[ f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2). \]

Expected value \( E[\hat{g}] \) if \( \nu^2 \hat{g} \) is discrete

then its expectation is
(weighted average) \( E(\bar{X}) = \sum_{i} \gamma_i \cdot \Gamma(\bar{X} = \gamma_i) \) if \( \gamma_i \) is the weight.

\[ E(\bar{X}) = \int_{-\infty}^{\infty} \gamma f_{\bar{X}}(\gamma) \, d\gamma = \mu \]

The variance of \( \bar{X} \) is given by \( \sigma^2 = E[ (\bar{X} - E(\bar{X}))^2 ] = V(\bar{X}) \) (spread) or the standard deviation (SD) of \( \bar{X} \) is \( \sigma = \sqrt{V(\bar{X})} \).

\[ \mu - \sigma \quad \mu \quad \mu + \sigma \]

68%
\[ SE_{\text{IID}}(\hat{\theta}) = \frac{0.19}{\sqrt{5}} = \sqrt{\frac{p(1-p)}{n}} \]

\[ SE(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{(0.18)(0.82)}{400}} = 0.019 = 1.9\% \]

\[ \hat{p} = 18\% \]

\[ SE(\hat{p}) = 1.9\% \]

\[ \hat{p} - 2SE \quad \hat{p} \quad \hat{p} + 2SE \]

\[ P = (0.14 \leq p \leq 0.22) \]

\[ \text{Density is repeated sampling of } \hat{p} \]

\[ 95\% \]

\[ P = \frac{\hat{p}}{\text{undefined}} \]