(1) [60 pts] Let \( X_1, \ldots, X_n \) be a random sample from the pdf
\[
f(x|\mu, \sigma) = \frac{1}{\sigma} \exp \left\{ -\frac{(x - \mu)}{\sigma} \right\} \quad \mu \leq x < \infty \quad 0 < \sigma < \infty
\]

(a) Find a minimal sufficient statistic for \((\mu, \sigma)\).
(b) Find \((\hat{\mu}, \hat{\sigma})\), the maximum likelihood estimator for \((\mu, \sigma)\).
(c) Is \(\hat{\mu}\) an unbiased estimator for \(\mu\)?

\[
(\alpha) \quad f(x|\mu, \sigma) = \left( \frac{1}{\sigma} \right)^n \exp \left\{ -\frac{1}{\sigma} \sum_{i=1}^{n} (x_i - \mu) \right\} \quad \mathcal{N}(\mu, \min_{\xi} x_i, y)
\]

So a candidate for the mass is \((x_0, \sum x_i)\)

It is indeed minimal because
\[
\frac{f(x|\mu, \sigma)}{f(x|\mu, \sigma)} \text{ is constant as function of } \mu, \sigma \text{ iff } (X_0, \sum_{i=1}^{n} \hat{\xi} x_i) = (y_0, \sum_{i=1}^{n} \hat{y}_i)
\]

(6) For the MLE, note that the likelihood is increasing.

So \(\hat{\mu} = x_0\), no matter what \(y\) is.

Also
\[
l(\mu, \sigma) = -n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^{n} x_i + \frac{n \mu}{\sigma}
\]

\[
\Rightarrow \frac{\partial}{\partial \sigma} = -\frac{n}{\sigma} + \frac{n \bar{x}}{\sigma^2} - \frac{n \hat{\mu}}{\sigma^2} = 0
\]

\[
\Rightarrow n = \frac{n \bar{x} - n \hat{\mu}}{\sigma} \Rightarrow \hat{\sigma} = \sqrt{\frac{\bar{x} - \hat{\mu}}{n - 1}} = \sqrt{\bar{x} - x_0}
\]
(c) \[ Y_i = \frac{X_i - \mu}{\sigma} \sim \text{Exp}(1) \]

\[
\Pr(Y_i > u) = \frac{\Pr(Y_i > u \forall i)}{n} = \frac{e^{-u} \times e^{-u} \times \cdots \times e^{-u}}{n \text{ times}} = e^{-nu}
\]

\[ Y_{(1)} \sim \text{Exp}(\frac{1}{n}) \]

\[ E(\hat{\mu}) \]

\[ E(X_{(1)}) = E(\mu + \sigma Y_{(1)}) = \mu + \sigma E(Y_{(1)}) \]

\[ = \mu + \sigma \frac{1}{n} \neq \mu \]

Hence, it is a biased estimator.
(2) [40 pts] Consider an infinite sequence of Bernoulli trials for which the parameter $0 \leq \theta \leq 1$ (the probability of success) is unknown, and suppose that sampling is continued until exactly $k$ successes have been obtained, where $k \geq 2$ is a fixed integer. Let $N$ denote the total number of trials that are needed to obtain the $k$ successes. Show that the estimator $\tilde{\theta} = (k-1)/(N-1)$ is the minimum variance unbiased estimator for $\theta$.

$$N \sim \text{Neg Bin} (k, \theta)$$

Note that

$$E(\tilde{\theta}) = \sum_{N=k}^{\infty} \frac{k-1}{N-1} \binom{N-1}{k-1} \theta^k (1-\theta)^{N-k}$$

$$\begin{align*}
&= \theta \sum_{N=k}^{\infty} \frac{(k-1)(N-1)!}{(N-1)(k-1)!(N-k)!} \theta^{k-1} (1-\theta)^{N-k} \\
&= \theta \sum_{N=k}^{\infty} \frac{(N-2)!}{(k-2)!(N-k)!} \theta^{k-1} (1-\theta)^{N-k} = \theta \\
& \uparrow \text{(Kernel of a Neg Bin (k-1, \theta)} \\
& \text{after a change of variables}
\end{align*}$$

Hence $\tilde{\theta}$ is unbiased.

Since $N$ is the complete sufficient statistic for this problem (the neg Binomial is part of the exponential family), this has to be the MVUE for $\theta$. 