Hypothesis testing

Power curve

\[ \beta(\theta) = P(\hat{\theta} \in R(\theta)) \]

(\theta) is the true value of the parameter

When \( \theta \in \Theta_0 \), \( \beta(\theta) \) is just the type I error rate.

When \( \theta \in \Theta_\theta \), \( 1 - \beta(\theta) \) is the type II error.

Therefore, as long as the sample size is fixed, there is no way to control both errors.

Example:

\[ X_1, \ldots, X_n \sim N(\mu, \sigma^2) \]

- \( \sigma^2 = 1 \) known.
- \( \mu \) is unknown.

\[ H_0: \mu = \mu_0 \quad \nu \leq \quad H_a: \mu \neq \mu_0 \]
LRT for this problem was derived last class.

Reject $|\bar{X} - \mu_0| > K$ for some $K$

For example, pick $K = 1.96 \frac{\sigma}{\sqrt{n}} \rightarrow$ sample size

$\beta(\mu): \Pr \left( |\bar{X} - \mu_0| > K \mid \text{Xi's have mean } \mu \right)$

$= 1 - \Pr ( |\bar{X} - \mu_0| < K \mid \mu )$

$= 1 - \Pr ( -K < \bar{X} - \mu_0 < K \mid \mu ) \quad \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$= 1 - \Pr ( \mu_0 - K < \bar{X} < \mu_0 + K \mid \mu )$

$= 1 - \Pr \left( \frac{\mu_0 - K - \mu}{\sigma/\sqrt{n}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{\mu_0 + K - \mu}{\sigma/\sqrt{n}} \right)$

$= 1 - \Phi \left( \frac{\mu_0 - K - \mu}{\sigma/\sqrt{n}} \right) + \Phi \left( \frac{\mu_0 + K - \mu}{\sigma/\sqrt{n}} \right)$
Level & size.

A test has size \( \alpha \) if

\[
\sup_{\theta \in \Theta} \beta(\theta) = \alpha.
\]

A test has level \( \alpha \) if

\[
\sup_{\theta \in \Theta_0} \beta(\theta) \leq \alpha.
\]

If the null is a point hypothesis

\[
\sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_0)
\]

Example:

\( X_1, \ldots, X_n \sim \text{Binomial}(n, \theta) \) \( n \geq 1 \)

\( H_0: \theta = \theta_0 \) vs \( H_a: \theta \neq \theta_0 \)

LRT is

Reject \( H_0 \) if

\[
\sum X_i > K_1 \text{ or } \sum X_i < K_2
\]
\(\theta_0 = \frac{1}{2} \quad m = 10\)

Can we design a size \(\alpha\) test? For any \(\alpha (\alpha = 0.05)\)

To have size \(\alpha\), we would need to pick \(k_1\) and \(k_2\) so that

\[\beta(\theta_0) = \underbrace{\Pr(\theta = \theta_0 | X < k_1)} + \underbrace{\Pr(\theta = \theta_0 | X > k_2)} = 0.05\]

What kind of values make sense for \(k_1\) and \(k_2\)? \(\Rightarrow\) Integer.

The question is, can I find integers \(k_1\) and \(k_2\) that satisfy this equation? Show it!!

The problem is that the distribution of \(X\) is discrete, so you have to live with a test of size \(\alpha\) instead.
p-values:

Another way to think about testing

Instead of computing a rejection region according to the size/level of the test that we want, we could compute

\[ P(T \text{ statistic is at least as extreme as observed } | H_0 \text{ is true}) = p-value. \]

Example:

\[ X_1, \ldots, X_n \sim N(\mu, \sigma^2)\]

\[ H_0: \mu = \mu_0 \quad H_1: \mu \neq \mu_0. \]

Test statistic is \[ T(X) = \frac{\bar{X} - \mu_0}{\sigma_0}. \]

\[ T(X) \text{ has some distribution under the null, } \mathcal{N}(0, 1). \]

\[ P(T(\bar{X}) > T(X)) = p-value \]

\[ \left\{ \text{Theoretical Sample} \right\} \xrightarrow{observed} \text{sample} \]
\[ T(\bar{x}) = |\bar{x} - \mu_0| \]

\[ \bar{x} - \mu_0 \sim N(0, \sigma^2 / n) \]

\[ Pr \left( |\bar{x} - \mu_0| > t(x) \right) = \]

\[ Pr \left( \frac{|\bar{x} - \mu_0|}{\sigma / \sqrt{n}} > \frac{t(x)}{\sigma / \sqrt{n}} \right) = Pr \left( 1.21 > \frac{|\bar{x} - \mu_0|}{\sigma / \sqrt{n}} \right) \]

\[ \Rightarrow z \sim N(0,1) \]

For example, if \( \sigma = 1 \), \( n = 10 \), \( \mu_0 = 0 \) and you observe \( \bar{x} = 1.6 \)

Then p-value = \( Pr \left( |1.21| > \frac{1.6}{1} \right) \)

If you reject \( H_0 \) when the p-value is less than \( \alpha \) then this is equivalent to a level/size \( \alpha \) test.