1. Excercise 1.2.8
Solution \( C_k, \ k = 1, 2, \ldots \) is a sequence of increasing sets.

(a) \( C_k = \{ x : 1/k \leq x \leq 3 - 1/k \}, \ k = 1, 2, 3, \ldots \)
As \( k \to \infty \), \( 1/k \to 0 \) and \( (3 - 1/k) \to 3 \).

\[
\lim_{n \to \infty} C_k = \bigcup_{k=1}^{\infty} C_k = (0, 3)
\]
Note that the two points 0 and 3 are not in any of \( C_k \) for \( k = 1, 2, 3, \ldots \). Therefore, these two points are not included in \( \lim_{n \to \infty} C_k \) above.

(b) \( C_k = \{ (x, y) : 1/k \leq x^2 + y^2 \leq 4 - 1/k \} \)
As \( k \to \infty \), \( 1/k \to 0 \) and \( (4 - 1/k) \to 4 \).

\[
\lim_{n \to \infty} C_k = \bigcup_{k=1}^{\infty} C_k = \{(x, y) : 0 < x^2 + y^2 < 4\}
\]

2. Excercise 1.2.9
Solution \( C_k, \ k = 1, 2, \ldots \) is a sequence of decreasing sets.

(a) \( C_k = \{ x : 2 - 1/k < x \leq 2 \} \)
As \( k \to \infty \), \( (2 - 1/k) \to 2 \). The l.h.s of the inequality is approaching 2 from left then.

By definition,

\[
\lim_{n \to \infty} = \bigcap_{k=1}^{\infty} C_k = \{2\}
\]
Since 2 is the only element that is in all \( C_k \) s.

(b) \( C_k = \{ x : 2 < x \leq 2 + 1/k \} \) As \( k \to \infty \), \( (2 + 1/k) \to 2 \). The r.h.s of the inequality is approaching 2 from right then.

By definition,

\[
\lim_{n \to \infty} = \bigcap_{k=1}^{\infty} C_k = \emptyset
\]
Since there is no single element could be found that is in all $C_k$ s. To see this, we could assume there exists a sufficiently small $\delta > 0$ such that $x = 2 + \delta$. Certainly, $x > 2$ must hold then. However, we could always find another sufficiently large $k$ such that $1/k < \delta$. Then it follows that $x = 2 + \delta > 2 + 1/k$, which contradicts our assumption.

(c) $C_k = \{(x, y) : 0 \leq x^2 + y^2 \leq 1/k\}$

As $k \to \infty$, $1/k \to 0$.

Note the right part of the inequalities above includes equality. Consequently, $0$ is the only element in the shrinking $C_k$ sequence.

Hence,

$$\lim_{n \to \infty} = \bigcap_{k=1}^{\infty} C_k = \{(0, 0)\}$$

3. Exercise 1.3.10

Solution

(a) Follow the hint, denote

$$C_i := \{\text{an exact match on the } i\text{th turn}\}$$

Also, by the general inclusion-exclusion formula, we could get

$$p_M = \Pr\{\text{at least one exact match}\}$$

$$= P(C_1 \cup C_2 \cup \ldots \cup C_{52})$$

$$= \binom{52}{1} P(C_i) - \binom{52}{2} P(C_i \cap C_j) + \binom{52}{3} P(C_i \cap C_j \cap C_k) - \ldots - \binom{52}{52} P(\cap_{i=1}^{52} C_i)$$

Further, define $p_2 = \binom{52}{2} P(C_i \cap C_j)$, $p_3 = \binom{52}{3} P(C_i \cap C_j \cap C_k)$, $\ldots$. Then,

$$p_M = p_1 - p_2 + p_3 - \ldots - p_{52}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots - \frac{1}{52!}$$

(\star)

as desired.
(b) Recall the Taylor expansion of $e^x$,

$$e^x = 1 + x + \frac{1}{2}x^2 + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

Evaluating the above equation at $x = -1$, we get

$$e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + \ldots$$

Then we could get

$$1 - e^{-1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots$$

Note that ($\star$) is similar as the above equation, then we reach

$$1 - e^{-1} \approx p_M$$

4. **Exercise 1.3.15**

**Solution**  The number of all possible outcomes: $\binom{8}{2}$

The number of possible outcomes of two red chips: $\binom{5}{2} \cdot \binom{6}{0}$

The number of possible outcomes of two blue chips: $\binom{3}{2} \cdot \binom{6}{0}$

The number of possible outcomes of two 1s: $\binom{2}{2} \cdot \binom{6}{0}$

The number of possible outcomes of two 2s: $\binom{2}{2} \cdot \binom{6}{0}$

The number of possible outcomes of two 3s: $\binom{2}{2} \cdot \binom{6}{0}$

Then,

$$\Pr \{ \text{2 chips randomly drawn have either same color or same number} \}$$

$$= \frac{\binom{5}{2} \cdot \binom{6}{0} + \binom{3}{2} \cdot \binom{6}{0} + 3 \cdot \binom{2}{2} \cdot \binom{6}{0}}{\binom{8}{2}}$$

5. **Exercise 1.4.8**

**Solution**  Let $D$ denote the event that a spring is defective; and I, II, III the events that
a spring is produced by Machine I, II and III, respectively. We know the following probabilities

\[ P(I) = 1\%, \quad P(D \mid I) = 30\% \]
\[ P(II) = 4\%, \quad P(D \mid II) = 25\% \]
\[ P(III) = 2\%, \quad P(D \mid III) = 45\% \]

(a) Randomly draw a spring, find the probability that is defective.

\[
P(D) = P(D \mid I) \cdot P(I) + P(D \mid II) \cdot P(II) + P(D \mid III) \cdot P(III)
\]
\[
= 0.3 \times 0.01 + 0.25 \times 0.04 + 0.45 \times 0.02
\]
\[
= 0.022
\]

(b) Given a randomly drawn spring is defective, what is the probability that is produced by Machine II?

\[
P(II \mid D) = \frac{P(D \mid II) \cdot P(II)}{P(D)}
\]
\[
= \frac{0.04 \times 0.25}{0.022}
\]
\[
= 5/11
\]

6. Excercise 1.4.19

**Solution** Define \( A := \{ \text{at least 4 draws to obtain a spade} \} \). Then we could have the following,

\[ A^c = \{ \text{at most 3 draws to obtain a spade} \}
\[ = \{ \text{obtain a spade at the 1}^{st} \text{ draw} \} + \]
\[ \{ \text{obtain a spade at the 2}^{st} \text{ draw} \} + \]
\[ \{ \text{obtain a spade at the 3}^{st} \text{ draw} \} \]
(a) Draw cards with replacement.

\[ P(A) = 1 - P(A^c) \]
\[ = 1 - \left( \frac{13}{52} + \frac{52 - 13}{52} \cdot \frac{13}{52} + \left( \frac{52 - 13}{52} \right)^2 \cdot \frac{13}{52} \right) \]
\[ = \frac{27}{64} \]

(b) Draw cards without replacement.

\[ P(A) = 1 - P(A^c) \]
\[ = 1 - \left( \frac{13}{52} + \frac{52 - 13}{52} \cdot \frac{13}{51} + \frac{52 - 13}{52} \cdot \frac{38}{51} \cdot \frac{13}{50} \right) \]
\[ = 1 - \left( \frac{13}{52} + \frac{3}{4} \cdot \frac{13}{51} + \frac{3}{4} \cdot \frac{38}{51} \cdot \frac{13}{50} \right) \]

7. Exercise 1.4.32

Solution Suppose that we could select \( p_1 \) and \( p_2 \) satisfies the following equation

\[ P(\text{zero hits}) = P(\text{one hit}) = P(\text{two hits}) \]

Note that by assuming independence, and plugging \( p_1, p_2 \) to the chain of equation above

\[
\begin{align*}
(1 - p_1)(1 - p_2) &= (1 - p_1)p_2 + (1 - p_2)p_1 \\
(1 - p_1)p_2 + (1 - p_2)p_1 &= p_1p_2
\end{align*}
\]

Solving for the above system, we could get

\[ p_1 = \frac{1}{3p_2}, \quad 3p_2^2 - 3p_2 + 1 = 0 \]

It then turns out that we couldn’t get a real root for \( p_2 \). Thus, it is not possible to select \( p_1, p_2 \) satisfying the condition.

8. Find pmf of \( Y = X^3 \), where

\[
p_X(x) = \begin{cases} 
\left(\frac{1}{2}\right)^2 & x = 1, 2, \ldots \\
0 & \text{otherwise}
\end{cases}
\]
**Excercise 1.6.8**

**Solution**  First, identify the support of $Y$, $S_Y = \{1, 8, 27, \ldots \}$.

\[
p_Y(y) = P(Y = y) = \sum_{x \in g^{-1}(y)} P(X = x) = \sum_{x \in g^{-1}(y)} p_X(x) = p_X\left(\sqrt[3]{y}\right) = \left(\frac{1}{2}\right)^{\sqrt[3]{y}}
\]

Hence, we could have the pmf of $Y$ as

\[
p_Y(y) = \begin{cases} 
\left(\frac{1}{2}\right)^{\sqrt[3]{y}} & \text{For } y \in S_Y \\
0 & \text{otherwise}
\end{cases}
\]

\[\square\]

9.  **Excercise 1.7.9**

**Solution**  Median of distribution.

(a)

\[
p(x) = \begin{cases} 
\frac{4!}{x!(4-x)!} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{4-x} & x = 0, 1, 2, 3, 4 \\
0 & \text{otherwise}
\end{cases}
\]

Computing $P(X \leq x)$ and $P(X < x)$ for $x = 1, 2, 3, 4$, it easy to see that

\[
P(X \leq 1) = p(0) + p(1) = \left(\frac{3}{4}\right)^3 + \frac{4!}{1! \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)} = \left(\frac{3}{4}\right)^4 = \frac{189}{256} > \frac{1}{2}
\]

\[
P(X < 1) = p(0) = \frac{81}{256} < \frac{1}{2}
\]

Then the median for this distribution is 0.
(b) 

\[ f(x) = \begin{cases} 
3x^2 & 0 < x < 1 \\
0 & \text{otherwise} 
\end{cases} \]

Since point probability of continuous r.v. is 0, then

\[ P(X \leq x) = P(X < x) \implies P(X \leq x) = 1/2 \]

Calculate the cdf,

\[
F(x) = P(X \leq x) = \int_0^x 3t^2 \, dt = \frac{3x^3}{3} = x^3
\]

Then, by solving \( x^3 = 1/2 \) yields the median for this distribution is \( \sqrt[3]{1/2} \).

(c) 

\[ f(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty \]

Calculate the cdf as

\[
F(X) = P(X \leq x) = \int_{-\infty}^x \frac{1}{\pi(1 + t^2)} \, dt = \frac{1}{\pi} \int_{-\infty}^x \frac{1}{1 + t^2} \, dt = \frac{1}{\pi} \left[ \arctan t \right]_{-\infty}^x = \frac{1}{\pi} \arctan x + \frac{1}{2}
\]

Then, by solving \( \frac{1}{\pi} \arctan x + \frac{1}{2} = \frac{1}{2} \) yields the median for this distribution is \( x = 0 \).
10. Excercise 1.7.21
Solution

\[ f(x) = \begin{cases} 
2xe^{-x^2} & 0 < x < \infty \\
0 & \text{otherwise}
\end{cases} \]

Let \( \mathcal{X}, \mathcal{Y} \) denote the support of the r.v. X and Y, respectively.
Note the transformation \( Y = g(X) = X^2 \) is monotone on \( \mathcal{X} \); the support transformation is then
\[ \mathcal{X} = \{ x : 0 < x < \infty \}, \quad \mathcal{Y} = \{ y : 0 < y < \infty \} \]
Identify the inverse as \( g^{-1}(y) = y^{\frac{1}{2}} \), for \( y \in \mathcal{Y} \).
Compute the Jacobian of transformation
\[ J = \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{2} y^{-\frac{1}{2}} \]
Now we could calculate the pdf of \( Y \) within the support \( \mathcal{Y} \)
\[ f_Y(y) = f_X(g^{-1}(y)) \cdot J = e^{-y} \]
Then, we could get the pdf of \( Y \) as
\[ f_Y(y) = \begin{cases} 
e^{-y} & 0 < y < \infty \\
0 & \text{otherwise}
\end{cases} \]

11. Excercise 1.8.10
Solution

\[ f(x) = \begin{cases} 
2x & 0 < x < 1 \\
0 & \text{otherwise}
\end{cases} \]
(a) $E[1/x]$

Compute the expectation directly as

$$E\left[\frac{1}{X}\right] = \int_0^1 \frac{1}{x} \cdot f(x) \, dx$$

$$= \int_0^1 \frac{1}{x} \cdot 2x \, dx$$

$$= 2$$

(b) Compute the cdf and pdf of $Y = 1/X$

Identify the inverse as $g^{-1}(y) = 1/y$ and $\mathcal{Y} = (1, \infty)$, also note that $g(\cdot)$ is monotonically decreasing.

Also get the cdf of $X$ as $F_X(x) = \int_0^x 2t \, dt = x^2$, then

$$F_Y(y) = 1 - F_X(g^{-1}(y)) = 1 - y^{-2}$$

Compute the Jacobian of transformation as $J = \left| \frac{d}{dy} \frac{1}{y} \right| = y^{-2}$, then the cdf is

$$f_Y(y) = \begin{cases} \frac{2}{y^3} & 1 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

(c) Calculate the expectation by definition

$$E_Y = \int_{-\infty}^{\infty} y \cdot f_Y(y) \, dx$$

$$= \int_{1}^{\infty} y \cdot 2y^{-3} \, dy$$

$$= -2 \cdot \left[ y^{-1} \right]_{1}^{\infty}$$

$$= 2$$