Exponential family

Consider the collection

\[ f(x|\theta) : \theta \in \Omega \] where

\[ f(x|\theta) = \begin{cases} \exp\left\{ \sum_{i=1}^{T} P_i(\theta) K_i(x) + S(\theta) + q(x) \right\} & x \in S \\ 0 & \text{otherwise} \end{cases} \]

This is said to belong to the exponential family.
Examples:

1) $X \sim \text{Gamma}(\alpha, \lambda)$ with $\alpha$ known and fixed

$$f(x) := \begin{cases} \frac{1}{\Gamma(\alpha) \lambda^\alpha} x^{\alpha-1} \exp\left\{- \frac{x}{\lambda}\right\} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \exp\left\{- \frac{x}{\lambda} + (\alpha-1) \log x - \alpha \log \lambda - \log \Gamma(\alpha) \right\} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$q(x) = (\alpha - 1) \log x$

$s(\theta) = -\alpha \log \lambda - \log \Gamma(\alpha)$

$p(\theta) = -\frac{1}{\lambda}$

$k(x) = x$
2) \( X \sim N(\mu, \sigma^2) \) with both \( \mu \) and \( \sigma^2 \) unknown.

\[
f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left\{-\frac{1}{2\sigma^2} (x - \mu)^2\right\}
\]

\[
= \exp\left\{-\frac{1}{2} \log 2\pi - \log \sigma - \frac{1}{2\sigma^2} (x^2 - 2\mu x + \mu^2)\right\}
\]

\[
\theta(x) = \frac{1}{2} \log 2\pi
\]

\[
S(\mu, \sigma^2) = -\log \sigma - \frac{\mu^2}{2\sigma^2}
\]

\[
P_1(\mu, \sigma^2) = -\frac{1}{2\sigma^2}
\]

\[
K_1(x) = x^2
\]

\[
P_2(\mu, \sigma^2) = \frac{\mu}{\sigma^2}
\]

\[
K_2(x) = x
\]

\[\Rightarrow \text{Yes, it is a member of the exponential family}\]
\[ f(x|\theta) = \begin{cases} \frac{1}{\theta} & x \in [0, \theta] \\ 0 & \text{otherwise} \end{cases} \]

\[ = \frac{1}{\theta} \mathbb{1}_{[0, \theta]}(x) = \frac{1}{\theta} \mathbb{1}_{b(c, x \in \theta < 1)} \]

The problem here is that the support depends on the parameter. We are going to exclude this type of distributions and talk about the regular exponential family.

This family satisfies all the regularity conditions we discussed when looking at MLE ... (generally speaking)
Properties

1) $E(X) \Rightarrow$ from $\theta \in S(\theta)$
2) $\text{Var}(X)$

$f(x_1, \ldots, x_n)$ if $X_i \text{ iid exponential family}$

$$= \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \exp\left\{ R(\theta \cdot K(x_i) + S(\theta) + g(x_i) \right\}$$

$$= \exp\left\{ R(\theta \cdot [\sum_{i=1}^{n} K(x_i)] + n \{ S(\theta) + \sum_{i=1}^{n} g(x_i) \right\}$$

Is there a sufficient statistic for this sample?

Yes $\sum_{i=1}^{n} K(x_i)$ is the sufficient stat.
For example, what is the sufficient stat.
for $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$

\[ \left( \frac{\sum_{i=1}^{n} X_i}{n}, \frac{\sum_{i=1}^{n} X_i^2}{n} \right) \]

\[ \left( \frac{\sum_{i=1}^{n} X_i}{n}, \frac{\sum_{i=1}^{n} X_i^2}{n} - n \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right)^2 \right) \]

This is also a Suf. Stat

How do you get the MLE for a member of the exponential family?

$$L(\theta) = R(\theta) \cdot \sum_{i=1}^{n} K(x_i) + S(\theta)$$

$$\frac{\partial L(\theta)}{\partial \theta} = \sum_{i=1}^{n} K(x_i) + \rho S'(\theta) = 0$$

In the univariate case solve

$$\sum_{i=1}^{n} K(x_i) = -\frac{s'(\theta)}{p'(\theta)}$$
\( x_1, \ldots, x_n \) is a sample from a member of the exponential family, let
\[
Y = \sum_{i=1}^{n} X_i \quad \text{sufficient statistic}
\]

1) The pdf for \( Y \) can be written as
\[
f_{Y,y}(y|\theta) = R(y) \exp \left\{ p(\theta) y + n \frac{S(\theta)}{p(\theta)} \right\}
\]
for some \( R(y) \) that does not depend on \( \theta \)

\( \Rightarrow \) This family turns out to be complete

2) \( E(Y) = -n \frac{S'(\theta)}{p'(\theta)} \)

3) \( \text{Var}(Y) = n \frac{1}{[p'(\theta)]^2} \left\{ p'(\theta) S''(\theta) - S'(\theta) p'(\theta) \right\} \)

If \( \theta \) is univariate