**Probability (a review)**

**Uncertainty.**

Experiment: any trial in which the outcome cannot be predicted.

**Probability:**

1) The average number of successes in a large number of repetitions (frequency interpretation).

2) Subjective chance of some future event happening (subjective interpretation).

Bayesian

\[
\text{Fair value of a bet}
\]

Outcome space \( \Rightarrow \) set of all possible outcomes of your experiment.

Event space \( \Rightarrow \) set of subsets of the outcome space that we want to assign probabilities to.

**Probability \( \Rightarrow \) a set function defined on the event space that has the following properties:**

- \( P(\emptyset) = 0 \)
- \( P(\Omega) = 1 \)
- If \( A_1, A_2, \ldots \) are disjoint sets then
  \[
  P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)
  \]
Set Theory:

Elements: $a \in A$ if $a$ belongs to the set $A$.

Subsets: $A \subseteq B$ is the set of all elements of $A$ that are also elements of $B$.

Unions: $A \cup B$ is the set that contains all elements that are in $A$ or $B$.

Intersection: $A \cap B$ is the set that contains all elements that are in $A$ and $B$.

$A \subseteq A \cup B$

$A \cap B \subseteq A$

Complements: $A^c$ is the set of all elements that are not in $A$.

Difference: $A \setminus B$ is the set of elements of $A$ that are not in $B$.

De Morgan's Law:

\[
\overline{(A \cup B)} = \overline{A} \cap \overline{B}
\]

\[
\overline{(A \cap B)} = \overline{A} \cup \overline{B}
\]
Properties of a probability function:

\[ P(A^c) = 1 - P(A) \]

Inclusion-Exclusion rule:
If \( A \) and \( B \) are disjoint \( A \cap B = \emptyset \) then \( P(A \cup B) = P(A) + P(B) \)

What if the intersection is not empty:
\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

What if you have 3 sets?
\[ P(A \cup B \cup C) = P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C)) \]

\[ = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap (B \cup C)) \]

\[ = P(A) + P(B) + P(C) - P(A \cap C) - P(A \cap B) - P(B \cap C) + P(A \cap B \cap C) \]

Boole's inequality:
\[ P(A_1 \cup A_2 \cup A_3 \cdots \cup A_n) \leq P(A_1) + P(A_2) + \cdots + P(A_n) \]

Bonferroni inequality:
\[ P(A \cap B) \geq P(A) + P(B) - 1 \]
How do we figure out probabilities in practice.

Symmetry leads us to think that all outcomes can be equally likely, and therefore to compute a probability we only need to be able to count.

- Throwing 2 dice:
  - Obtaining a sum equal to 4 can happen in 3 ways: 1+3 or 2+2
  - Obtaining a sum equal to 5 can also happen in two ways: 1+4 or 2+3.

The error is that symmetry is true only for the ordered pairs!!!

**Counting (combinatorics):**

- **Multiplication rule.**
  - If Event A can happen in \( n \) ways and exp. B can have \( m \) different outcomes and there is no relationship between them then A and B together can happen in \( n \times m \) different forms.

- **Permutations:**
  - Number of ways in which I can order \( n \) distinct elements:
    \[ n \cdot (n-1) \cdot (n-2) \cdots 1 = n! \]
Conditional probabilities.

<table>
<thead>
<tr>
<th>Cancer</th>
<th>Y</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>N</td>
<td>100</td>
<td>500</td>
</tr>
</tbody>
</table>

What is the probability that somebody suffers from cancer? 
\[
\frac{40}{640}
\]

What is the probability that a smoker has cancer?
\[
\frac{30}{130}
\]

\[P(A|B) = \frac{P(A \cap B)}{P(B)}\]
Combinatorial numbers

- How many ways can I pick \( m \) elements from a set that has a total of \( n \) if I care about the order in which I am drawing the subjects?

\[
30 \times 29 \times 28 \times 27 \times 26 = \frac{30!}{(30-s)!} \frac{n!}{(n-m)!}
\]

- What if I do not care about the order.

\[
\frac{n!}{m! \cdot (n-m)!}
\]

Homework: for a group of 30 people, what is the probability that no two of them have the same birthday.
Independence.

A and B are independent if knowledge about one does not affect the probability of the other happening.

\[ P(A \mid B) = P(A) \quad \text{and} \quad P(B \mid A) = P(B) \]

\[ \Rightarrow P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \Rightarrow \quad P(A \mid B) = \frac{P(A \cap B) \cdot P(B)}{P(B)} \]

\[ = \frac{P(A) \cdot P(B)}{P(B)} \]

\[ \text{If independent} \]

Bayes' theorem:

\[ P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)} \]

Assume that \( B_1, \ldots, B_n \) is a partition of the space \( \Omega \)

\[ \bigcup_{i=1}^{n} B_i = \Omega \]

\[ B_i \cap B_j = \emptyset \quad \text{if} \quad i \neq j \]

Then

\[ P(B_i \mid A) = \frac{P(A \cap B_i) \cdot P(B_i)}{\sum_{i=1}^{n} P(A \cap B_i) \cdot P(B_i)} \]

\[ \Rightarrow \sum_{i=1}^{n} P(A \cap B_i) \cdot P(B_i) = P(A) \]