\[ \hat{\lambda}(\mathbf{X}, \mathbf{y}) \text{ is an estimator of } \lambda \text{ for the random sample} \]

- \[\hat{\lambda}\] is a random quantity.
- \[E(\hat{\lambda}) = \lambda\] if \[\lambda\] is a constant quantity.
- \[\text{Bias}(\hat{\lambda}) = E(\hat{\lambda}) - \lambda\]
- \[\text{Unbiased estimator} \iff E(\hat{\lambda}) = \lambda\]
- \[\mathbf{X}_1, \ldots, \mathbf{X}_n \sim \text{Exp}(\lambda) \implies E(\hat{\lambda}) = \lambda\]
- \[\hat{\lambda} = \bar{X}_n\] is an unbiased estimator of \(\lambda\)
- \[\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i\]
- \[\bar{X}_n \overset{\text{d}}{\rightarrow} \lambda\] as \(n \rightarrow \infty\)

\[
\begin{align*}
\text{Variance of } \hat{\lambda} & = \text{Var}(\hat{\lambda}) = \frac{1}{n^2} \text{Var}(\bar{X}_n) = \frac{1}{n^2} \text{Var}(\sum_{i=1}^{n} X_i/n) \\
& = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(X_i) = \frac{\lambda}{n} \\
\end{align*}
\]
\begin{align*}
\text{MSE}(\theta) &= \mathbb{E}(\hat{\theta}^2 - \theta^2) \\
\text{Var}(\hat{\theta}) &= \mathbb{E}((\hat{\theta} - \mathbb{E}(\hat{\theta}))^2)
\end{align*}

\text{MSE}(\hat{\theta}^2 - \theta^2) = \mathbb{E}((\hat{\theta}^2 - \theta^2)^2)

\text{Bias}^2 + \text{Var}(\hat{\theta}) + \mathbb{E}(\hat{\theta}^4) - \theta^4

\mathbb{E}(\theta^2) = \frac{1}{n} \sum_{i=1}^{n} \theta_i^2

\mathbb{E}(\hat{\theta}^2) = \frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_i^2

\text{Var}(\hat{\theta}) = \frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_i^2 - \left( \frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_i \right)^2
Concluding

Case 2: May be any way that the estimator $\hat{\theta}$ is "nearby" to the true value $\theta$.

The reader should note there are multiple ways in which $\hat{\theta}$ can be "nearby."

Some comments on likelihood.

If $X \sim \text{Normal}(\mu, \sigma^2)$ is a normal variable, defined

we can probably agree with any that the

peaks $\approx 6$ for another normal variable if

$\mu = 0$

In reality, we may worry about $\mu$ and

we consider all $\mu$ values in a single peak.

Conclusion: As above, $\hat{\theta}$ is unbounded

in $\theta = \frac{1}{\sigma^2} \log \left( \frac{1}{n} \sum \theta \right)$ $\sigma > 0$
\[ \text{MSE}(\theta) = \text{MSE}(\theta) \cdot \frac{Z^2}{\nu c} \]

\[ \text{MSE}(\theta) = \left[ \frac{(\theta_{\text{true}} - \theta)^2}{\nu c} \right] \cdot \frac{Z^2}{\nu c} \]

\[ = \frac{1}{\nu c} \cdot \frac{Z^2}{\nu c} \]

\[ \text{MSE}(\theta) = \frac{Z^2}{\nu c} \]

\[ \text{MSE}(\theta) \quad \text{for some} \quad \theta_{\text{true}} \]

Note: Some thresholds might be better than others, just for some \( \theta_{\text{true}} \).
Example:
\[ x_0, y_0 \in \mathbb{R}^n \] with \[ \sigma^2 \text{ known} \].

Consider \[ \mathcal{N}(x, \Sigma) \]. Is this estimator unbiased?

\[ P(\{ |\mu - \lambda| < \epsilon \}) \]

\[ \mu - \lambda \sim N(0, \epsilon^2) \]

\[ P(\{ |\mu - \lambda| < \epsilon \}) = \frac{1}{\sqrt{2\pi\epsilon^2}} \int_{-\epsilon}^{\epsilon} e^{-\frac{x^2}{2\epsilon^2}} dx \]

\[ = 1 - e^{-\frac{\epsilon^2}{2}} \]

\[ \Rightarrow \text{Conjecture } (1) \]

\[ \Rightarrow 1 \]
Theorem: If \( \lambda_n \) is a sequence of estimators and
\[
\lim_{n \to \infty} \text{E}(\hat{\theta}) = \theta
\]
and
\[
\lim_{n \to \infty} \text{Var}(\hat{\theta}) = 0
\]
Then \( \hat{\theta} \) is a consistent estimator.

Example:
\( \lambda_n = \hat{\theta} \) is a sequence of estimators
\( \frac{\hat{\theta}}{\hat{\theta}} \) converges
\( \frac{\hat{\theta}}{\hat{\theta}} \)
\( \text{E}(\hat{\theta}) = \frac{\hat{\theta}}{\hat{\theta}} \) is unbiased
\( \text{Var}(\hat{\theta}) = \frac{\hat{\theta}}{\hat{\theta}} \)
\( \lim_{n \to \infty} \text{E}(\hat{\theta}) = \theta \)
\( \lim_{n \to \infty} \text{Var}(\hat{\theta}) = 0 \)

Estimator is consistent.
\[ \text{as discussed earlier} \]
\[ \Delta \approx \frac{3}{4} \]

\[ \text{as discussed previously} \]
\[ \Delta \approx \frac{3}{4} \]

\[ \text{as discussed above} \]
\[ \Delta \approx \frac{3}{4} \]

\[ \text{as discussed in examples} \]
\[ \Delta \approx \frac{3}{4} \]

\[ \text{as previously discussed} \]
\[ \Delta \approx \frac{3}{4} \]

\[ \sum \text{as previously discussed} \]
\[ \Delta \approx \frac{3}{4} \]

\[ \text{as previously discussed} \]
\[ \Delta \approx \frac{3}{4} \]

\[ \text{as previously discussed} \]
\[ \Delta \approx \frac{3}{4} \]

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\[ \text{as previously discussed} \]
\[ \Delta \approx \frac{3}{4} \]
Theorem 4.25

$X \to Y$

$Z \to Z'$

$X \to Z$

$2 \to X$

Convergence in distribution:

What is the behavior of the cumulative distribution function as $n \to \infty$?

$X_n, X_{n+1}, \ldots$ is a sequence of random variables (not necessarily in the same probability space) and define $F_n, F_{n+1}, \ldots$ as their cumulative distribution functions.

1. $F_n$ converges in distribution to $F$.
2. $\lim_{n \to \infty} F_n(x) = F(x)$ for all continuity points of $F$.
3. Further convergence of the cumulative distribution function.


\[
\frac{\sqrt{\frac{E(A)}{2}}}{\frac{\sigma}{\sqrt{N}}} \rightarrow \text{Normal}
\]

\[
\text{Chi-square: } \chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}
\]

\[
\text{Fisher's exact test: } P = \sum_{r=0}^{\min(n_1, n_2)} \binom{n_1}{r} \binom{n_2}{n-r} \left( \frac{1}{n_1 n_2} \right)
\]

\[
\text{Logistic regression: } \log \left( \frac{p}{1-p} \right) = \beta_0 + \beta_1 x
\]

\[
\text{Multiple regression: } y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \epsilon
\]

\[
\text{Multiple comparison: } q = \text{Adjusted } t \text{-value}
\]

\[
\text{Correlation: } r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}
\]