AMS 205 – Homework 4

due in class, Tuesday October 26

1. Let $X$ be the mean of a random sample of size 5 from a normal distribution with $\mu = 0$ and $\sigma^2 = 125$. Determine $c$ so that $P(X < c) = 0.90$.

2. Let $X_1, \ldots, X_{25}$ and $Y_1, \ldots, Y_{25}$ be two independent random samples from two normal distributions, $N(0, 16)$ and $N(1, 9)$ respectively. Let $\overline{X}$ and $\overline{Y}$ denote the corresponding sample means. Compute $P(\overline{X} > \overline{Y})$.
   \textit{Hint:} Note that $P(\overline{X} > \overline{Y}) = P(\overline{X} - \overline{Y} > 0)$, and determine the distribution of $\overline{X} - \overline{Y}$.

3. Let $S^2$ be the variance of a random sample of size 6 from the normal distribution $N(\mu, 12)$. Find $P(2.30 < S^2 < 22.2)$.

4. Let $X_1, X_2, \ldots, X_n$ have a multivariate normal distribution, where $\mu$ is the matrix of means and $V$ is the positive definite covariance matrix. Let $Y = c'X$ and $Z = d'X$ where $X' = [X_1 \ X_2 \ \ldots \ X_n]$, $c' = [c_1 \ c_2 \ \ldots \ c_n]$, and $d' = [d_1 \ d_2 \ \ldots \ d_n]$ are real matrices.
   \begin{enumerate}
   \item Find the joint MGF $M(t_1, t_2) = E[\exp(t_1 Y + t_2 Z)]$ and see that $Y$ and $Z$ have a bivariate normal distribution. (Note: $Y$ and $Z$ are scalars.)
   \item Prove that $Y$ and $Z$ are independent if and only if $c'V d = 0$.
   \end{enumerate}

5. If $X_i \sim Bin(n_i, p)$ for $i = 1, 2, \ldots, k$, and the $X_i$’s are mutually independent, find the distribution of $Y = \sum_{i=1}^{k} X_i$.

6. Let $X_n$ have a gamma distribution with parameters $\alpha = n$ and $\beta$, where $\beta$ is not a function of $n$. Let $Y_n = \frac{X_n}{n}$. Find the limiting distribution of $Y_n$ (as $n \to \infty$).