Hypothesis testing.

\[ H_0: \theta \in \Theta_0 \quad \text{vs} \quad H_e: \theta \notin \Theta_0 \]

1) Statistic that "measures difference between null and the data"

2) Rejection region for the null hypothesis

Ex \[ X_1, \ldots, X_n \sim N(M, \sigma^2) \quad \sigma^2 \text{ known.} \]
\[ H_0: M = M_0 \quad \text{vs} \quad H_e: M \neq M_0 \]

Likelihood ratio tests

\[ -\Lambda (X) : \frac{\sup_{\Theta \in \Theta_0} L(\Theta)}{\sup_{\Theta \in \Theta} L(\Theta)} \quad \Rightarrow \text{statistic.} \]

\( L(\Theta) \) is the likelihood (the same function you used to get MLEs)

In our example

\[ f(x_1, \ldots, x_n | \mu) = \prod_{i=1}^{n} \left( \frac{1}{\sqrt{2\pi}\sigma} \right) \exp \left\{ -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right\} \]

\[ = \left( \frac{1}{\sqrt{2\pi}n} \right)^n \exp \left\{ -\frac{1}{2\sigma^2} \sum (x_i - \mu)^2 \right\} \]
\[ \Lambda(x) = \frac{L(m_0)}{L(x)} \]

has to be greater than 0 and less than 1

For some \( k \), we should fail to reject if \( \Lambda(x) > k \)

Reject if \( \Lambda(x) \leq k \)

This is the likelihood ratio test for this problem.
\[ L(\mu) = \exp \left\{ -\frac{1}{2\sigma^2} \sum (x_i - \mu)^2 \right\} \]

What is the MLE for this problem. (for \( \mu \))

\[ \hat{\mu} = \frac{1}{n} \sum x_i = \bar{x} \quad \text{obtained by solving} \quad \frac{dL}{d\mu} = 0 \]

\[-L(\mu) = \sup_{\theta \in \Theta_0} \frac{L(\theta)}{\sup_{\theta \in \Theta_0} L(\theta)} \]

Since \( \Theta_0 = \{ \mu_0 \} \) is a single point

the numerator \( \sup_{\theta \in \Theta_0} L(\theta) = L(\mu_0) \)

\[ L(\mu) = \exp \left\{ -\frac{1}{2\sigma^2} \sum (x_i - \mu_0)^2 \right\} \]

Since \( \Theta_0^c = \mathbb{R} \setminus \{ \mu_0 \} \) the denominator

\[ \sup_{\theta \in \Theta_0^c} L(\theta) = L(\hat{\mu}) = L(\bar{x}) \]

\[ L(\bar{x}) = \exp \left\{ -\frac{1}{2\sigma^2} \sum (x_i - \bar{x})^2 \right\} \]
\[ L(m) = \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x} + \bar{x} - m)^2 \right\} \]

\[ = \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x} + \bar{x} - m)^2 \right\} \]

\[ \sum_{i=1}^{n} (x_i - \bar{x} + \bar{x} - m)^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 + 2 \sum_{i=1}^{n} (x_i - \bar{x})(\bar{x} - m) \]

\[ + \sum_{i=1}^{n} (\bar{x} - m)^2 \]

\[ = \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - m)^2 + 0 \]

Note that \( \sum (x_i - \bar{x}) = \sum x_i - n\bar{x} = \sum x_i - 2\bar{x} \bar{x} = 0 \)

\[ L(m) = \exp \left\{ -\frac{n}{2\sigma^2} (\bar{x} - m)^2 \right\} \]
Reject $H_0$ if $L(x) < c$

$$\exp\{-\frac{1}{2\sigma^2} (\bar{x} - M_0)^2\} < c$$

$$-\frac{1}{2\sigma^2} (\bar{x} - M_0)^2 < \log c$$

$$\frac{1}{2}\sigma^2 \bar{x}^2 > -\frac{1}{2\sigma^2}\log c$$

$$|\bar{x} - M_0| > \sqrt{-\frac{1}{2\sigma^2}\log c}$$

The LRT is equivalent to reject if

$$|\bar{x} - M_0| > k$$

How to pick $k$?
Power function:

\[
\beta(\theta) = \Pr(\text{Rejecting } H_0 \mid \theta)
\]

- If the true value of the parameter is \( \theta \)

  - When \( \theta \in \Theta_0 \), \( \beta(\theta) \) gives you the probability of type I errors
  - When \( \theta \notin \Theta_0 \), \( \beta(\theta) \) gives you 1 minus the probability of type II error

\[
\text{Type I error rate} + \text{Type II error rate} \neq 1
\]
What do we want for error rates?
So would like to minimize both types of errors. But there is a trade-off
Minimizing type I error, implies increasing type II error on vice versa.

The most common approach to deal with this is to control one of the two error
rates (type I error typically) and live with whatever type II error that implies.

Usually, we want to design the test, so that type I error is 0.05
0.10
0.01
In the previous example.

Reject \( 1X - \mu_0 > K \)

What value should \( K \) take if I want a test with a 0.05 probability of a type I error?

\[
\Pr (\text{type I error}) = \Pr (\text{Rejecting null} \mid \text{Ho true}) = 0.05
\]

\[
\Pr (\text{Rejecting null} \mid \text{Ho true}) = \Pr (1X - \mu_0 > K) \quad x \sim N(\mu_0, \sigma^2)
\]

If the mean is really \( \mu_0 \)

\[
X - \mu_0 \sim N(0, \sigma^2)
\]

\[
1 - \Pr (-K < X - \mu < K) = 1 - \Pr \left( \frac{\sqrt{n}}{\sigma} K < \frac{(X - \mu_0)}{\frac{\sigma}{\sqrt{n}}} \right)
\]
\[
1 - P_{\eta} \left( -\frac{\kappa \sqrt{n}}{\sigma} < \frac{n}{\sigma}(\bar{X} - \mu) < \frac{\kappa \sqrt{n}}{\sigma} \right) = 0.05
\]

\[
P_{\eta} \left( -\frac{\kappa \sqrt{n}}{\sigma} < \frac{n}{\sigma}(\bar{X} - \mu) < \frac{\kappa \sqrt{n}}{\sigma} \right) = 0.95
\]

\[
\mathcal{N}(0,1)
\]

\[
\frac{\kappa \sqrt{n}}{\sqrt{\sigma}} = 1.96 \quad \Rightarrow \quad \kappa = 1.96 \frac{\sigma}{\sqrt{n}}
\]
Example: \( X_1, \ldots, X_n \sim \text{Ber}(\theta) \) \( \theta \): probability of success

\( H_0: \theta = \frac{1}{2} \) \( \iff \theta \neq \frac{1}{2} \)

\[
f(x_i | \theta) = \theta^{x_i} (1-\theta)^{1-x_i}
\]

\[
f(x_1, \ldots, x_n | \theta) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i} = \theta^{\sum x_i} (1-\theta)^{n-\sum x_i}
\]

\[
l(\theta) = (\sum x_i) \log \theta + (n-\sum x_i) \log (1-\theta)
\]

\[
\frac{dl}{d\theta} = \frac{\sum x_i}{\theta} - \frac{(n-\sum x_i)}{1-\theta} = 0
\]

\[
\Rightarrow \frac{\sum x_i}{\hat{\theta}} = \frac{n-\sum x_i}{1-\hat{\theta}} \Rightarrow \sum x_i - \hat{\theta} \sum x_i = \frac{\hat{\theta} \sum x_i - (\sum x_i)^2}{\hat{\theta} (1-\hat{\theta})}
\]

\[
\Rightarrow \hat{\theta} = \frac{1}{n} \sum x_i \Rightarrow \text{let} \sum x_i = S
\]

\[
= \frac{S}{n}
\]
$$L(x) = \sup_{\theta \in \Theta_0} L(\theta)$$

$$L(\theta) = \theta^{\sum x_i} (1-\theta)^{n-\sum x_i}$$

$$\sup_{\theta \in \Theta_0} L(\theta) = L(\hat{\theta}) = \left( \frac{1}{2} \right)^{\sum x_i} \left( \frac{1}{2} \right)^{n-\sum x_i}$$

$$= \left( \frac{1}{2} \right)^s \left( \frac{1}{2} \right)^{n-s} = \left( \frac{1}{2} \right)^n$$

$$\sup_{\theta \in \Theta_0^c} L(\theta) = L(\hat{\theta}) = L\left( \frac{s}{n} \right) = \left( \frac{s}{n} \right)^s \left( 1 - \frac{s}{n} \right)^{n-s}$$

Reject $H_0$ if

$$\frac{\left( \frac{1}{2} \right)^n}{\left( \frac{s}{n} \right)^s \left( 1 - \frac{s}{n} \right)^{n-s}} < C \text{ for some } C$$
You can show that

\[
\frac{\left(\frac{1}{2}\right)^n}{\left(\frac{s}{n}\right)^s \left(1-\frac{s}{n}\right)^{n-s}} < c \quad \text{if and only if}
\]

\[s \geq k_1\]

or

\[s \leq k_2\]

If we want to find a test with type I error rate of 0.05 we need to pick \(k_1\) and \(k_2\) so that

\[\Pr(S \geq k_1 \mid \theta = \frac{1}{2}) \leq 0.025\]

\[\Pr(S \leq k_2 \mid \theta = \frac{1}{2}) \leq 0.025\]

\[S \mid \theta = \frac{1}{2} \sim \text{Bin}(n, \frac{1}{2})\]

Usually we make the interval symmetric

\[k_2 = n - k_1\]
This is not the only way to construct tests. As long as you have a statistic whose distribution depends only on the parameter you are testing, you could use that as your starting place.

\[ X_1, \ldots, X_n \sim N(\mu, \sigma^2) \quad \sigma^2 \text{ known.} \]

\[ H_0: \mu = \mu_0 \quad vs \quad H_0: \mu \neq \mu_0 \]

\[ T(x_1, \ldots, x_n) = \# \text{ of observations that are below } \mu_0 \]

\[ T \sim \text{Bin} \left(n, \Phi \left( \frac{\mu_0 - \mu}{\sigma} \right) \right) \]

\[ \Rightarrow \Pr (X_i \leq \mu_0) \]

Under the null \( \mu = \mu_0 \) \( \Phi(0) = \frac{1}{2} \)

Since, as long as \( \mu \) is the median, \( T \sim \text{Bin}(n, \frac{1}{2}) \) under the null \( \Rightarrow \) non-parametric
How do we pick among multiple tests that have the same type I error? We could choose the test with the lowest type II error (largest power).

\( X_1, \ldots, X_n \sim N(\mu, \sigma^2) \) \( \sigma^2 \) known.

\( H_0: \mu = \mu_0 \) vs \( H_a: \mu \neq \mu_0 \)

\text{LRT: } \text{Reject } H_0 \text{ if } |\bar{X} - \mu_0| > \frac{1.965}{\sqrt{n}}

This test has type I error rate of 0.05 by construction.

What is the type II error associated with this test? Since \( \mu \neq \mu_0 \) is \( H_a \), it has more than one point in it, and you need a whole function.

\( \Pr( \text{fail to reject null} | H_a \text{ true} ) \)

\[ \Pr \left( |\bar{X} - \mu_0| < \frac{1.965}{\sqrt{n}} \mid X_i \sim N(\mu, \sigma^2) \right) \quad (\text{for every possible } \mu) \]
\[
\Pr \left( -1.96 \frac{\sigma}{\sqrt{n}} < \bar{X} - \mu_0 < 1.96 \frac{\sigma}{\sqrt{n}} \mid X_i \sim N(\mu, \sigma^2) \right) \\
= \Pr \left( \mu_0 - \mu - 1.96 \frac{\sigma}{\sqrt{n}} < \bar{X} - \mu < \mu_0 - \mu + 1.96 \frac{\sigma}{\sqrt{n}} \right) \\
= \Pr \left( \frac{\sqrt{n}}{\sigma} (\mu_0 - \mu) - 1.96 < \frac{\sqrt{n}}{\sigma} (\bar{X} - \mu) < \frac{\sqrt{n}}{\sigma} (\mu_0 - \mu) + 1.96 \right) \\
\sim N(0, 1) \\
= \Phi \left( \frac{\sqrt{n}}{\sigma} (\mu_0 - \mu) - 1.96 \right) - \Phi \left( \frac{\sqrt{n}}{\sigma} (\mu_0 - \mu) + 1.96 \right)
\]