SOLUTION SUBSPACES

Consider the homogeneous system
\[ \begin{align*}
x_1 + 3x_2 - 15x_3 + 7x_4 &= 0 \\
x_1 + 4x_2 - 19x_3 + 10x_4 &= 0 \\
2x_1 + 5x_2 - 26x_3 + 11x_4 &= 0
\end{align*} \]

The reduced-row echelon form of the coefficient matrix \( A \) is
\[
\begin{bmatrix}
1 & 0 & -3 & -2 \\
0 & 1 & -4 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Hence,
- **leading variables**: \( x_1, x_2 \)
- **free variables**: \( x_3, x_4 \)

We set \( x_3 = s \), \( x_4 = t \). With back substitution,
\[
\begin{align*}
x_2 &= 4s - 3t \\
x_1 &= 3s + 2t
\end{align*}
\]

Therefore, a typical solution is
\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s + 2t \\ 4s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\((\mathbb{R}^3, +, \cdot)\) is a vector space

(i) \( u + v = v + u \) (commutativity)
(ii) \( u + (v + w) = (u + v) + w \) (associativity)
(iii) \( u + 0 = u = 0 + u \) (zero element)
(iv) \( u + (-u) = 0 = (-u) + u \) (additive inverse)
(v) \( r \cdot (u + v) = r \cdot u + r \cdot v \) (distributivity)
(vi) \( (r + s) \cdot u = r \cdot u + s \cdot u \)
(vii) \( r \cdot (s \cdot u) = (rs) \cdot u \)
(viii) \( 1 \cdot u = u \) (multiplicative identity)
Property 1: If \( B \) is obtained from \( A \) by multiplying a single row (or column) of \( A \) by a constant \( k \), then \( \det B = k \cdot \det A \).

Property 2: If \( B \) is obtained from \( A \) by interchanging two rows (or columns), then \( \det B = -\det A \).

Property 3: If two rows (or two columns) of \( A \) are identical, then \( \det A = 0 \).

Property 4: If \( A_1, A_2, B \) have the same rows except for one, and that row of \( B \) is the sum of the corresponding rows of \( A_1 \) and \( A_2 \), then \( \det B = \det A_1 + \det A_2 \).

Property 5: If \( B \) is obtained by adding a constant multiple of a row (or column) of \( A \) to another row (or column) of \( A \), then \( \det B = \det A \).

Property 6: The determinant of a triangular matrix is equal to the product of its diagonal elements.

Property 7: \( \det (A^T) = \det A \).

Linearly Independent Sets, Spanning Sets & Basis

Let \( V \) be an \( n \)-dimensional vector space.

Let \( S \) be a subset of \( V \).

Then:

(a) if \( S \) is linearly independent and consists of \( n \) vectors, then \( S \) is a basis for \( V \).

(b) if \( S \) spans \( V \) and consists of \( n \) vectors, then \( S \) is a basis for \( V \).

(c) if \( S \) is linearly independent, then \( S \) is contained in a basis for \( V \).

(d) if \( S \) spans \( V \), then \( S \) contains a basis for \( V \).
**Bases for Solution Spaces**

\[ V = \{ x \in \mathbb{R}^n : Ax = 0 \} \]

\( Ax = 0 \). We use elementary row operations to put \( A \) in echelon form. Suppose that the leading variables are the first \( r \) variables, so the remaining \( k = n - r \) variables are free.

\[
\begin{align*}
&b_1 x_1 + b_{12} x_2 + \ldots + b_{1r} x_r + \ldots + b_{1n} x_n = 0 \\
&b_{22} x_2 + \ldots + b_{2r} x_r + \ldots + b_{2n} x_n = 0 \\
&\vdots \\
&b_{rr} x_r + \ldots + b_{rn} x_n = 0
\end{align*}
\]

\( 0 = 0 \) of these.

We set \( x_{r+1} = t_1, \ldots, x_k = t_k \).

By back substitution,

\[
\begin{align*}
&x_1 = c_1 t_1 + \ldots + c_r t_r \\
&\vdots \\
&x_r = c_{r1} t_1 + \ldots + c_{rk} t_k \\
&x_{r+1} = t_1 \\
&\vdots \\
&x_n = t_k
\end{align*}
\]

\[ x = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ \vdots & \vdots & & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rk} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nk} \end{bmatrix} \begin{bmatrix} t_1 \\ \vdots \\ t_k \end{bmatrix} = t_1 \begin{bmatrix} v_1 \\ \vdots \\ v_1 \end{bmatrix} + \ldots + t_k \begin{bmatrix} v_k \\ \vdots \\ v_k \end{bmatrix}
\]

\( v_1, \ldots, v_k \) is a basis for \( V = \{ x \in \mathbb{R}^n : Ax = 0 \} \).

By the way, if \( k = 0 \) (no free variables), then \( V = \{ 0 \} \).
Row space of an echelon matrix

\[
A = \begin{bmatrix}
1 & -3 & 2 & 5 & 3 \\
0 & 0 & 1 & -4 & 2 \\
0 & 0 & 0 & 1 & 7 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Row vectors are:
\[r_1 = (1, -3, 2, 5, 3)\]
\[r_2 = (0, 0, 1, -4, 2)\]
\[r_3 = (0, 0, 0, 1, 7)\]
\[r_4 = (0, 0, 0, 0, 0)\]

We want to show that \(\{r_1, r_2, r_3\}\) are linearly independent.

\[c_1 r_1 + c_2 r_2 + c_3 r_3 = 0\]

\[c_1 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\]

\[\Rightarrow c_4 = 0\]
\[c_2 = 0\]
\[c_3 = 0\]

This works for any echelon matrix.

Column space of an echelon matrix

\[
E = \begin{bmatrix}
1 & 2 & 1 & 3 & 2 \\
0 & 1 & -3 & 5 & -4 \\
0 & 0 & 0 & 1 & -7 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\(e_1, e_2, e_4\) correspond to the leading entries.

Also \(\text{Col}(E) \subseteq \{(x_1, x_2, x_3, 0) \in \mathbb{R}^4\} \cong \mathbb{R}^3\)

Note that
\[a_1 e_1 + a_2 e_2 + a_4 e_4 = \begin{bmatrix} a_1 + 2a_2 + 3a_4 \\ a_2 + 5a_4 \\ a_4 \end{bmatrix}\]

So \(a_1 e_1 + a_2 e_2 + a_4 e_4 = 0 \iff a_1 = a_2 = a_4 = 0\)

Therefore \(e_1, e_2\) and \(e_4\) are linearly independent, and consequently they form a basis of \(\text{col}(E)\).

So the column rank of \(E\) is 3.

This works for any echelon matrix.
INNER PRODUCT

An inner product on a vector space $V$ is a function that to each pair of vectors $u, v \in V$ associates a scalar $\langle u, v \rangle$ such that:

(i) $\langle u, v \rangle = \langle v, u \rangle$
(ii) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
(iii) $\langle cu, v \rangle = c \langle u, v \rangle$
(iv) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \iff u = 0$

Given a 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we can find:

- Two distinct real eigenvalues, each corresponding to a single eigenvector:
  
  \[
  A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}, \quad \lambda_1 = -2, \quad \lambda_2 = 3, \quad v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
  \]

- A single real eigenvalue corresponding to a single eigenvector:
  
  \[
  A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}, \quad \lambda = 2, \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
  \]

- A single real eigenvalue corresponding to two linearly independent eigenvectors:
  
  \[
  A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda = 1, \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
  \]

- Two complex conjugate eigenvalues corresponding to complex conjugate eigenvectors:
  
  \[
  A = \begin{bmatrix} 0 & 8 \\ -2 & 0 \end{bmatrix}, \quad \lambda_1 = -4i, \quad \lambda_2 = 4i, \quad v_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix}^T, \quad v_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}^T
  \]
Computing the eigenspaces of $A$

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{bmatrix}$$

Expanding the first row,

$$|A - \lambda I| = (4-\lambda) \left( -\lambda (3-\lambda) + 2 \right) - (-2) \left[ 2(3-\lambda) + 2 \right] + 1 \left[ (-4+2\lambda) \right] = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = (\lambda-2)^2 (\lambda-3)$$

Therefore, eigenvalues of $A$ are $\lambda = 2$ and $\lambda = 3$.

For the eigenvectors: let's write $(A - \lambda I) \mathbf{v} = 0$

**Case $\lambda = 2$**

$$\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff 2x - 2y + z = 0$$

$$\mathbf{v}_1 = [1, 1, 0]^T$$

$$\mathbf{v}_2 = [-4, 0, 2]^T$$

**Case $\lambda = 3$**

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x - 2y + z = 0$$

$$\mathbf{v}_3 = [1, 1, 1]^T$$

Existence & Uniqueness of Solutions

Suppose $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on a rectangle $R$ that contains $(a, b)$ in its interior.

Then, for some open interval $I$ that contains $a$, the problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$

has one and only one solution defined on $I$. 

$$x + y = 0$$

$$-x + z = 0$$

$$x = y$$

$$y + z = 0$$

$$x = y$$
Linear First-Order Equations & Integrating Factors

Linear First-Order Differential Equation
\[ \frac{dy}{dx} + P(x) \cdot y = Q(x) \]

Integrating Factor
\[ e^{\int P(x) \, dx} \]

1. We multiply the diff. eq. by the integrating factor to get:
\[ e^{\int P(x) \, dx} \frac{dy}{dx} + P(x) \cdot e^{\int P(x) \, dx} \cdot y = Q(x) \cdot e^{\int P(x) \, dx} \]

\[ D(x) \cdot \frac{dy}{dx} = \int (Q(x) \cdot e^{\int P(x) \, dx}) \, dx + C \]

2. Therefore, integrating with respect to \( x \),
\[ y(x) \cdot e^{\int P(x) \, dx} = \int (Q(x) \cdot e^{\int P(x) \, dx}) \, dx + C \]

3. And finally,
\[ y(x) = e^{-\int P(x) \, dx} \left[ \int (Q(x) \cdot e^{\int P(x) \, dx}) \, dx + C \right] \]

Mixture Problems
A tank of water with salt.

- \( x(t) \) - amount of salt at time \( t \)
- \( r_i \) - liters per second flowing into the tank
- \( c_i \) - concentration of salt flowing into the tank
- \( r_0 \) - liters per second flowing out of the tank
- \( c_0 \) - concentration of salt in water flowing out of the tank

In time \( \Delta t \),
\[ r_i \cdot c_i \cdot \Delta t \] grams of salt into the tank
\[ r_0 \cdot c_0 \cdot \Delta t \] grams of salt out of the tank

Note \( c_0 = \frac{x(t)}{V(t)} \) (\( V(t) \) is the volume, \( V(t) = V_0 + (r_i - r_0) t \))

Therefore,
\[ \Delta x = r_i c_i \Delta t - r_0 c_0 \Delta t \]
\[ \frac{dx}{dt} = \frac{r_i c_i - r_0 c_0}{V_0 + (r_i - r_0) t} \]

First-Order Linear Equation
Substitution Methods & exact equations

\[ \frac{dy}{dx} = F(ax+by+c), \quad v = ax + by + c \quad \frac{dv}{dx} = a + bF(v) \]

Homogeneous equations:

\[ \frac{dy}{dx} = F \left( \frac{y}{x} \right), \quad v = \frac{y}{x} \quad x\frac{dv}{dx} = F(v) - v \]

Bernoulli equations:

\[ \frac{dy}{dx} + P(x)y = Q(x)y^n, \quad v = y^{1-n} \quad \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x) \]

Reducible second-order eqs:

\[ F(x,y,y',y'') = 0 \text{ without } x \text{ or without } y \]

\[ F(x,y',y'') = 0 \quad p = \frac{dy}{dx} \quad F(x,p,p') = 0 \]

\[ F(y,y',y'') = 0 \quad p = \frac{dy}{dx} \quad F(y,p,p,p_{pp}) = 0 \]

Criterion for exactness

\[ \det \begin{vmatrix} M & N \end{vmatrix} dy = 0. \] Assume that \( M, N \) are continua and have continuous first-order partial derivatives in the open rectangle \( R \). Then the differential equation is exact in \( R \) if and only if \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \)

"Nth"-order linear differential equations

\[ y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_n(x)y = 0 \]

Superposition of solutions for homogeneous equations

If \( y_1, \ldots, y_n \) are \( n \) solutions on the interval \( I \), then

\[ c_1y_1 + \ldots + c_n y_n, \quad c_1, \ldots, c_n \in IR \]

is also a solution on \( I \).

Existence and uniqueness

Consider \[ y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_n(x)y' + P_0(x)y = f(x) \]. Assume that \( P_0, \ldots, P_n \) and \( f \) are continuous on the open interval \( I \) containing \( a \). Then, for any \( n \) numbers \( b_0, \ldots, b_{n-1} \), the differential equation has a solution \( y \) on \( I \) that satisfies

\[ y(a) = b_0, \quad y'(a) = b_1, \ldots, \quad y^{(n-1)}(a) = b_{n-1} \]

Linear dependence of functions

The functions \( f_1, \ldots, f_n \) are linearly dependent on the interval \( I \) if there exist constants \( c_1, \ldots, c_n \) not all zero such that

\[ c_1f_1 + \ldots + c_nf_n = 0 \text{ on } I \]

that is

\[ c_1f_1(x) + \ldots + c_nf_n(x) = 0, \text{ for all } x \in I. \]
Let \( f_1, \ldots, f_n \) be \((n-1)\)-times differentiable functions. Their Wronskian is the \( n \times n \) determinant

\[
W = \begin{vmatrix}
    f_1 & f_2 & \cdots & f_n \\
    f'_1 & f'_2 & \cdots & f'_n \\
    \vdots & \vdots & \ddots & \vdots \\
    f^{(n-1)}_1 & f^{(n-1)}_2 & \cdots & f^{(n-1)}_n \\
\end{vmatrix}
\]

The Wronskian of \( n \) linearly dependent functions is 0 (Easy to prove!)

General solutions of homogeneous equations

Let \( y_1, \ldots, y_n \) be linearly independent solutions on \( I \) of a homogeneous \( n \)-th order linear differential equation. Then, any solution of the equation can be expressed as

\[ c_1 y_1 + \ldots + c_n y_n \]

for certain constants \( c_1, \ldots, c_n \in \mathbb{R} \).

Nonhomogeneous equations

Consider the equation

\[ y^{(n)} + P_1(x) y^{(n-1)} + \ldots + P_n(x) y + P_0(x) = f(x) \]

Let \( y_p \) be a particular solution on \( I \). Let \( y_1, \ldots, y_n \) be solutions (linearly independent) of the associated homogeneous equation. Then, any solution of the equation can be expressed as

\[ c_1 y_1 + \ldots + c_n y_n + y_p \]

for certain constants \( c_1, \ldots, c_n \in \mathbb{R} \).
**METHOD OF UNDETERMINED COEFFICIENTS**

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(x) \]

Assume \( f(x) \) is a linear combination of products of functions of the following types:

1. Polynomial in \( x \)
2. Exponential function \( e^{\alpha x} \)
3. \( \cos kx \) and \( \sin kx \)

(E.g., \( f(x) = (3 - 4x^2) e^{5x} - 4x^4 \sin 10x \))

Suppose that no term in \( f(x) \) or any of its derivatives appears satisfies the associated homogeneous equation.

Then

1. Take a trial solution \( y_p \) which is a linear combination of all linearly independent such terms and their derivatives.

2. Determine the unknown coefficients by substituting \( y_p \) into the nonhomogeneous eq.

\[ y_p(x) = s \left[ (A_0 + A_1 x + \ldots + A_n x^n) e^{\alpha x} \cos kx + (B_0 + B_1 x + \ldots + B_n x^n) e^{\alpha x} \sin kx \right] \]

where \( s \) is the smallest positive integer such that no term in \( y_p \) duplicates a term in the homogeneous solution.

**METHOD OF UNDETERMINED COEFFICIENTS**

(duplication case)

* First version cannot be applied: some of the terms involved in \( f(x) \) and its derivatives do not satisfy the homogeneous equation.

Here is what we should do: if \( f(x) \) is a linear combination of terms of the form

\[ P_m(x) e^{\alpha x} \cos kx \quad \text{and} \quad P_n(x) e^{\alpha x} \sin kx \]

then we take as trial solution

\[ y_p(x) = s \left[ (A_0 + A_1 x + \ldots + A_n x^n) e^{\alpha x} \cos kx + (B_0 + B_1 x + \ldots + B_n x^n) e^{\alpha x} \sin kx \right] \]

where \( s \) is the smallest positive integer such that no term in \( y_p \) duplicates a term in the homogeneous solution.
**Free undamped motion**

\[ c = 0 \text{ in } mx'' + cx' + kx = F(t) \]

\[ F(t) = 0 \]

That is, \( mx'' + kx = 0 \).

Define \( w_0 = \sqrt{\frac{k}{m}} \). Then we can write

\[ x'' + w_0^2 x = 0 \]

The **general solution** of this equation is of the form

\[ x(t) = A \cos w_0 t + B \sin w_0 t \]

\[ \star \] Let's do a nice trick!

\[ C = \sqrt{A^2 + B^2} \]

\[ \alpha = \arctan (A, B) \]

Then

\[ \cos \alpha = \frac{A}{C} \quad \text{and} \quad \sin \alpha = \frac{B}{C} \]

We rewrite the solution \( x(t) \) as

\[ x(t) = C \cdot \left( \frac{A}{C} \cos w_0 t + \frac{B}{C} \sin w_0 t \right) = \]

\[ = C \cdot (\cos \alpha \cos w_0 t + \sin \alpha \sin w_0 t) = \]

\[ x(t) = C \cdot \cos (w_0 t - \alpha) \]
Free, clamped motion

\[ F(t) = 0 \text{ in } mx'' + cx' + kx = F(t) \]

That is, \[ mx'' + cx' + kx = 0 \]

Define \( w_0 = \sqrt{\frac{k}{m}} \) and \( p = \frac{c}{2m} > 0 \). Then

\[ x'' + 2px' + w_0^2 x = 0 \]

The characteristic equation has roots

\[ r^2 + 2pr + w_0^2 = 0 \]

\[ r = -p \pm \sqrt{(p^2 - w_0^2)} \]

If \( p^2 - w_0^2 < 0 \) (i.e., \( c^2 < 4km \)), then we have two distinct complex eigenvalues \( -p \pm i\sqrt{w_0^2 - p^2} \). Then

\[ x(t) = e^{-pt} (A \cos wt + B \sin wt) \]

where \( w_1 = \sqrt{w_0^2 - p^2} \).

Using same trick as before, we can rewrite this as

\[ x(t) = C e^{-pt} \cos (w_1 t - \alpha) \]

Again \( x(t) \to 0 \) as \( t \to \infty \)

If \( p^2 - w_0^2 > 0 \) (i.e., \( c^2 > 4km \)), then we have two distinct real roots \( r_1, r_2 \), with \( r_1, r_2 < 0 \)

\[ x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]

Therefore \( x(t) \to 0 \) as \( t \to \infty \)

If \( p^2 - w_0^2 = 0 \) (i.e., \( c^2 = 4km \)), then we have a repeated real root \( r = -p, 0 \)

\[ x(t) = e^{-pt} (c_1 + t c_2) \]

Again \( x(t) \to 0 \) as \( t \to \infty \)
Undamped forced motion

\[ mx'' + kx = F_0 \cos(\omega t) \]

\[ \text{EXTERNAL FREQUENCY} \]

Solutions of associated homogeneous equation are

\[ x = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \]

Assume \( \omega = \omega_0 \)

We try \( x_p = A \cos(\omega t) \) and substitute, we get

\[ A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega^2 - \omega_0^2} \]

Do general solution is

\[ x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0/m}{\omega^2 - \omega_0^2} \cos(\omega t) \]

**Resonance**

Note that when \( \omega \rightarrow \omega_0 \), \( A \rightarrow \infty \)

Let's solve the equation when \( \omega = \omega_0 \).

[Duplication: \( \cos(\omega_0 t) \) is a solution of the associated homogeneous equation.]

We try \( x_p = t(A \cos(\omega_0 t) + B \sin(\omega_0 t)) \), yielding

**System of Linear Differential Equations**

\[
\begin{align*}
\frac{d^2}{dt^2}x_1 &= P_{11}(t)x_1 + \cdots + P_{1n}(t)x_n + f_1(t) \\
\frac{d^2}{dt^2}x_2 &= P_{21}(t)x_1 + \cdots + P_{2n}(t)x_n + f_2(t) \\
&\vdots \\
\frac{d^2}{dt^2}x_n &= P_{n1}(t)x_1 + \cdots + P_{nn}(t)x_n + f_n(t)
\end{align*}
\]

**Existence and Uniqueness of Solutions**

Suppose that the functions \( P_{ij} \), \( f_i \) are continuous on the open interval \( I \) containing \( a \). Then, for the initial conditions \( x_1(a) = b_1, \ x_2(a) = b_2, \ldots, \ x_n(a) = b_n \) there exists one and only one solution of (*) satisfying them.
What about the following?

\[
P(t) = \begin{bmatrix}
P_1(t) & \cdots & P_m(t) \\
\vdots & \ddots & \vdots \\
P_m(t) & \cdots & P_n(t)
\end{bmatrix}, \quad f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}
\]

And

\[
x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}
\]

Our system can be rewritten now as

\[x' = P(t) \cdot x + f(t)
\]

And all we know from Chapter 5 (principle of superposition, the Wronskian, general solution), pretty much works here.

**General Solution of Homogeneous Systems**

let \(x_1(t), \ldots, x_n(t)\) be a linearly independent solution of \(x' = P(t) \cdot x\) on \(I\), with \(P(t)\) continuous.

Then any solution of the equation is of the form

\[c_1 x_1(t) + \ldots + c_n x_n(t)\]

for some constants \(c_1, \ldots, c_n \in \mathbb{R}\).

**Observation:**

Let \(X(t) = [x_1(t), \ldots, x_n(t)]\), the matrix whose columns are the solutions \(x_1(t), \ldots, x_n(t)\).

Then,

\[c_1 x_1(t) + \ldots + c_n x_n(t) = X(t) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\]

So, if we want to solve

\[
\frac{dx}{dt} = P(t) \cdot x, \quad x(a) = b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}
\]

all we have to do is solve

\[
X(a) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}
\]
**The Wronskian**

\[ W(t) = W(x_1(t), \ldots, x_n(t)) = \begin{vmatrix} x_1(t) & \cdots & x_n(t) \end{vmatrix} \]

Matrix whose columns are the vectors \( x_1(t), \ldots, x_n(t) \)

Suppose \( x_1(t), \ldots, x_n(t) \) are solutions of \( x' = P(t) x \) on \( I \). Assume \( P(t) \) is continuous. Then

(i) if \( x_1, \ldots, x_n \) are linearly dependent on \( I \), then \( W = 0 \) on all \( I \).

(ii) if \( x_1, \ldots, x_n \) are linearly independent on \( I \), then \( W \neq 0 \) on all \( I \).

---

**Principle of Superposition**

\[ \frac{dx}{dt} = P(t) x \quad (\text{homogeneous equation because } f(t) = 0). \]

Let \( x_1(t), \ldots, x_n(t) \) be \( n \) solutions of the homogeneous equation on \( I \). Then for any \( c_1, \ldots, c_n \) constants,

\[ c_1 x_1(t) + \ldots + c_n x_n(t) \]

is a solution on \( I \).

[Note that \( x_1, \ldots, x_n \) are vector-valued.]

---

**Linearly (In)dependent Functions**

\( x_1(t), \ldots, x_n(t) \) are linearly dependent on \( I \) if there exist constants \( c_1, \ldots, c_n \) not all zero such that

\[ c_1 x_1(t) + c_2 x_2(t) + \ldots + c_n x_n(t) = 0 \text{ on } I. \]

Otherwise, they are linearly independent on \( I \).
Jordan normal form

So that you know...

"every n x n matrix has n linearly independent generalized eigenvectors."

If A (n x n matrix) has s linearly independent eigenvectors $v_1, \ldots, v_s$, then A is similar to a block-diagonal matrix of the Jordan normal form

$$J = \begin{bmatrix}
J_1 & & \\
& J_2 & \\
& & \ddots & 0 \\
& & & J_s
\end{bmatrix}$$

where each submatrix $J_i$ is of the form

$$J_i = \begin{bmatrix}
\lambda_i & 1 & & \\
& \lambda_i & 1 & \\
& & \ddots & 1 \\
& & & \lambda_i
\end{bmatrix}$$

where $\lambda_i$ is the eigenvalue corresponding to $v_i$.

If $\Theta_i$ is of size k x k, it corresponds to a length k chain of generalized eigenvectors based on the ordinary eigenvector $v_i$.

Arranging all generalized eigenvectors in the proper order, one gets Q x n matrix such that

$$Q^{-1} A Q = J$$

($J$ is unique, except for reordering of blocks)

- Any chain of generalized eigenvectors is linearly independent.
- If two chains of generalized eigenvectors are based on linearly independent eigenvectors, then the union of the chains is linearly independent.
LAPLACE TRANSFORMS

A function \( f \) is of exponential order as \( t \to \infty \)

\[ |f(t)| \leq M e^{ct} \quad t \geq T \quad (M, c, T > 0) \]

Theorem (existence of Laplace transforms)

If \( f \) is piecewise continuous, \( t \geq 0 \), and of exponential order as \( t \to \infty \), then \( \mathcal{L}(f) \) exists if \( s > c \).

Additionally, \( \lim_{s \to \infty} \mathcal{L}(f) = 0 \).

This limits the class of functions that can be Laplace transforms.

Theorem (uniqueness of inverse Laplace transforms)

Assume \( f \) and \( g \) satisfy hypotheses above. If

\[ \mathcal{L}(f) = \mathcal{L}(g) \quad \text{for all} \ s > c \]

then

\[ f(t) = g(t) \quad \text{whenever on} \ [0, \infty) \] both \( f \) and \( g \) are continuous.

Transforms of Derivatives

\[ \mathcal{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-1)}(0) - f^{(n)}(0) \]

\[ \mathcal{L}(f'(t)) = s F(s) - f(0) \]

Transforms of Integrals

\[ \mathcal{L} \left( \int_0^t f(t) \, dt \right) = \frac{F(s)}{s} \]

Translation on the \( s \)-Axis

\[ \mathcal{L}(e^{at} f(t)) = F(s-a) \]

Translation on the \( t \)-Axis

\[ \mathcal{L}(u(t-a) f(t-a)) = e^{-as} F(s) \]

Differentiation of Transforms

\[ \mathcal{L}(tf(t)) = -F'(s) \]

Integration of Transforms

\[ \mathcal{L} \left( \frac{f(t)}{t} \right) = \int_0^\infty F(s) \, ds \]
THE CONVOLUTION PROPERTY

Given \( f(t), g(t) \), define

\[
(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau
\]

If \( f, g \) are piecewise continuous and of exponential order when \( t \to \infty \), then

\[
\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)
\]

PARTIAL FRACTION DECOMPOSITION

If \( \mathcal{L}(f)(s) \) is less than that of \( \mathcal{L}(g)(s) \)

1st step: Factorize the denominator \( A(s) \) into

- linear factors: \((s-a)^n\)
- irreducible quadratic: \((s-a)^2 + b^2\)

2nd step:

a) For linear factor partial fractions

\[
\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \frac{A_3}{(s-a)^3} + \cdots + \frac{A_n}{(s-a)^n}
\]

b) For quadratic factor partial fractions

\[
\frac{A_1s+B_1}{(s-a)^2 + b^2} + \frac{A_2s+B_2}{(s-a)^2 + b^2} + \cdots + \frac{A_ns+B_n}{(s-a)^2 + b^2}
\]

TRANSFORMS OF PERIODIC FUNCTIONS

Let \( f \) be piecewise continuous and \( f(t+p) = f(t) \).

Then,

\[
\mathcal{L}(f)(s) = \frac{1}{1-e^{sp}} \int_0^P e^{-st} f(t)\,dt, \quad s > 0.
\]