2.5  The derivative of the exponential function

2.5.1  Case Study: The IQ test

IQ tests are standardized tests designed in the late 19th and early 20th century to assess a person’s “intelligence”. Modern IQ tests measure several factors which are related to intelligence, namely logical reasoning, math skills, language abilities, spatial relations skills, knowledge retained and the ability to solve novel problems.

The result of these tests is a single numeric score. The test is designed so that the distribution of test results (given a large enough sample of test-takers) looks like the following graph:

![IQ Test分布图](image)

The function describing this type of curve is one of the most commonly-used functions in all sciences. It goes by a few different names:

In this case study, we will study the properties of this function, and try to find out how to model it mathematically to reproduce the IQ test results.

2.5.2  Mathematical corner: The Normal Distribution (the Gaussian function)

Definition:
2.5.3 Case Study: The IQ test, part II

We would now like to find out which values of \( m \) and \( s \) should be used to fit the IQ test data. Let’s graph \( f(x) \) for different values of \( m \) and \( s \): what do you notice?

Can we prove that this is indeed the case? In order to check on idea number 1, we could use Calculus to find where the maximum of \( f(x) \) is. In order to do so, we need to know more about the derivative of the exponential function.

2.5.4 Mathematical corner: Derivative of exponential functions

Derivative of the natural exponential function:

Derivative of exponentials of functions:

Proof:

Examples:

- What is the derivative of \( h(x) = e^{3x} \)

- What is the derivative of \( h(t) = e^{at} \) where \( a \) is constant
• What is the derivative of \( h(x) = 3e^{x^2+1} \).

• What is the derivative of \( h(y) = 2e^{\sin(y)} \).

• What is the derivative of \( h(x) = a^x \).

2.5.5 Case Study: The IQ test, part III

Based on this, we can now find out more about \( f(x) \), and begin to prove some of our ideas. Let’s calculate and study \( f'(x) \):

We have established the first important fact about Gaussian functions:

Unfortunately, we can’t really differentiate between very narrow Gaussians, and very thick Gaussians using this method, since their corresponding signs tables would be exactly the same. In order to make progress, it’s useful to notice that there are other significant points on the graph of \( f(x) \):

How can we characterize such points from a mathematical point of view?
2.6 Higher-order derivatives and their graphical properties

2.6.1 Mathematical corner: Higher order derivatives and curvature

Higher order derivatives are simply successive derivatives of the same function. Hence, we can define for example the “second derivative”, the “third derivative”, etc.: 

**SECOND DERIVATIVE:**

**THIRD DERIVATIVE:**

**N-TH DERIVATIVE:**

**EXAMPLES:**

- $f(x) = (2x + 1)^3$
  - $f'(x) =$
  - $f''(x) =$
  - $f^{(3)}(x) =$
  - $f^{(4)}(x) =$
  - $f^{(n)}(x) =$

- $f(x) = e^x$
  - $f'(x) =$
  - $f''(x) =$
  - $f^{(3)}(x) =$
  - $f^{(4)}(x) =$
  - $f^{(n)}(x) =$
The most commonly used higher-order derivative is the second derivative. Since it is the derivative of the derivative:

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We can therefore use this information to be even more precise in our graphs of functions.

**Example:** \( f(x) = (x + 2)(x^2 - 2x + 3) \)

- Signs table for \( f(x) \)

- Signs table for \( f'(x) \)
• Signs table for $f''(x)$

• Graph of the function:

2.6.2 Case Study: The IQ test, part IV

Using this information, we can now characterize the location of the inflection point in the Gaussian function:
Using this, we can now draw a signs table for $f''(x)$, and sketch an annotated graph for $f(x)$.

This effectively proves our theory! Indeed,

- For large $s$:
- For small $s$:

Looking at the graph for the IQ test curve, it looks like:

Based on this, we can for example find out:

- The probability of having an IQ of 130, which is “2 sigma” above the mean is :

- The probability of having an IQ of 85, which is “1 sigma” below the mean is :

### 2.6.3 Mathematical Corner: Properties of a Normal Distribution

Based on all the work done above, we can now summarize the general properties of a Normal Distribution function: