EXAM 2
16 February 2011

Instructions

- Please turn your OFF your cell phone, iPod, etc..
- There are 4 questions worth a total of 40 points. 100\%=40 points.
- Please write neatly and show all your work. You do not need to simplify your answers, unless a numerical value is called for.
- No notes or books.
- NO calculators, NO PDAs, etc.
- Take your time, and check your answers.

Use the Method of Thinking!! And joy of the day to you

NAME:  Key

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Problem 1. Chops: Each question is worth 1 point. **No partial credit given.**

A) If \( f(x,y) = \ln(5x^4 + y^2) \) what is \( f_x \)?
\[
\frac{1}{5x^4 + y^2} \cdot 20x^3
\]
B) If \( f(x,y) = \ln(5x^4y^2) \) what is \( f_y \)?
\[
\frac{2}{y}
\]
C) If \( f(x,y) = e^{x^4y^2} \) what is \( f_x \)?
\[
\exp \left[ 5x^4y^2 \right] 20x^3y^2
\]

The loot, measured in pounds Sterling, obtained when spending \( t \) minutes looting a ship is \( L(t) = 120 \frac{t}{t+30} \).

D) What are the units of 100 on the right hand side?

**Pounds Sterling**

F) How long must be spent looting to acquire two thirds of the stuff on the ship?
\[
\frac{t}{t+30} = \frac{2}{3} \implies t = \frac{2}{3} (t+30) \implies t = 60 \text{ minutes} \quad \text{no units, no credit}
\]

G) Find the first three derivatives of \( f(x) = \frac{1}{x+2} \) without using the quotient rule.
\[
f(x) = (4+z)^{-1} \quad f' = -2(4+z)^{-2} \quad f'' = -2 \cdot 2(4+z)^{-3} \quad f''' = -6(4+z)^{-4}
\]

J) If \( f(x,y) = 6\ln[(x + 2y)^{-2}] \) what is \( f_{xyx} \)?
\[
f_x = \frac{-12}{x+2y} \quad f_{xy} = \frac{2y}{(x+2y)^2} \quad f_{xyx} = \frac{-4y}{(x+2y)^3}
\]

K) If \( f(x,y) = 4^{xy} \) what is \( f_y \). It may be helpful to recall that: \( 4^z = e^{\ln(4^z)} \)
\[
4^{xy} = e^{xy \ln(4)} \quad \text{so} \quad \frac{\partial}{\partial y}(4^{xy}) = \ln(4) e^{xy \ln(4)} = \ln(4) 4^{xy}
\]

ok
L) If \( f(x, y) = 2xe^{xy} \) what is \( f_y \)?

\[
f_y = 2x^2 e^{xy}
\]

M) If \( g(w, z) = w\ln(4z) \) what is \( g_z \)?

\[
g_z = \frac{w}{z}
\]
Problem 2 A) (5 pts) Find the critical point(s) of \( f(x, y) = x^2 + 3y^2 + 4x - 9y + 17 \).

\[
\begin{align*}
  f_x &= 2x + 4 \\
  f_y &= 6y - 9 \\
  f_{xx} &= 2 \\
  f_{yy} &= 6 \\
  f_{xy} &= 0
\end{align*}
\]

\[
\begin{align*}
  f_x &= 0 \Rightarrow x = -2 \\
  f_y &= 0 \Rightarrow y = \frac{3}{2}
\end{align*}
\]

\((-2, \frac{3}{2})\) is the only critical point.

2 pts

2 pts

B) (5 pts) Find and classify the critical points of \( f(x, y) = (y^2 - 4)(e^x - 1) \)

\[
\begin{align*}
  f_x &= (y^2 - 4)e^x \\
  f_y &= 2y(e^x - 1) \\
  f_{xx} &= e^x(y^2 - 4) \\
  f_{yy} &= (e^x - 1) \\
  f_{xy} &= 2ye^x
\end{align*}
\]

\[
\begin{align*}
  f_x &= 0 \Rightarrow y = \pm 2 \\
  f_y &= 0 \Rightarrow x = 0
\end{align*}
\]

C.P.S. at \((0, 2), (0, -2)\)

At either,

\[
D = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} - (\pm 2)^2 < 0
\]

\(\Rightarrow \) neither max nor min (or saddle)

1 pt.
Problem 3 A manufacturer produces two goods, A and B, with the joint cost function
\[ c(q_A, q_B) = -8 - 2q_A^2 + 3q_Aq_B + 30q_A + 12q_B + 0.5q_B^2. \]

The demand functions for the products are \( p_A = 35 - 2q_A^2 + q_B \) and \( p_B = 20 - q_B + q_A \).

A) (2 pts) What is the profit function?
\[
\Pi(q_A, q_B) = q_A \left( 35 - 2q_A^2 + q_B \right) + q_B \left( 20 - q_B + q_A \right) - \left[ -8 - 2q_A^2 + 3q_Aq_B + 30q_A + 12q_B + \frac{1}{2}q_B^2 \right]
\]

\[
\Pi = \left\{ \begin{array}{l}
35q_A - 2q_A^3 + q_Aq_B + 20q_B - q_B^2 + q_Aq_B + 8 + 2q_A^3 - 3q_Aq_B - 30q_A - 12q_B - \frac{1}{2}q_B^2 \\
5q_A - \frac{1}{2}q_A^2 + q_Aq_B + 8q_B - q_B^2 + 8
\end{array} \right\}
\]

B) (8 pts) What level of production maximizes profit?
\[
\frac{\partial \Pi}{\partial q_A} = 5 - q_A - q_B \Rightarrow \text{These } = 0 \Rightarrow q_A + q_B = 5 \Rightarrow q_A^* = 3
\]
\[
\frac{\partial \Pi}{\partial q_B} = -q_A + 8 - 2q_B
\]

\[
\frac{d^2 \Pi}{d q_A^2} = -1 \quad \text{2 pts}
\]
\[
\frac{d^2 \Pi}{d q_B^2} = -2 \quad \text{2 pts}
\]
\[
\frac{d^2 \Pi}{d q_A d q_B} = -1 \quad \text{2 pts}
\]

\[ D = \begin{bmatrix} -1 & -2 \end{bmatrix} \begin{bmatrix} -1 \end{bmatrix} > 0 \]
\[ \frac{d^2 \Pi}{d q_A^2} < 0 \]
\[ \text{Max} \]
\[ \text{2 pts} \]
Problem 4 A consumer has allotted $I$ dollars a month to pay for gasoline $g$, and natural gas, $n$, to heat her studio apartment. Gasoline costs $3.00$ a gallon and natural gas costs $0.5$ per therm. Her utility function for these two items is 

$$ U(g, n) = 3\ln(g) + 2\ln(n) - 1 $$

A) (1 pt) What is the relationship between her allocation and the number of gallons of gasoline and therms of natural gas that she uses? 

$$ \frac{3}{2} g + \frac{1}{2} n = \frac{I}{p^T} $$

B) (2 pts) Suppose that she decides to set $n = 1$ and just bundle up for the winter. Sketch $U(g, 1)$. Hint: If you memorized the figure from the practice exam and just reproduce it here you will be wrong, so think first.

$$ U(g, 1) = 3\ln(g) + 2\ln(1) \cdot 1 - 1 = 3\ln(g) - 1 $$

C) (2 pts) Interpret your sketch from part B.

1. No gas consumption $\Rightarrow$ negative utility (1 pt)
2. Consumption $\uparrow$ utility at a declining rate (1 pt)

D) (3 pts) What are the rates of consumption of gasoline and natural gas that optimize her utility?

$$ \hat{U}(g, n, \lambda) = 3\ln(g) + 2\ln(n) - 1 - \lambda \left[ 3g + \frac{1}{2} n - I \right] $$

$$ \hat{U}_g = \frac{3}{g} - 3\lambda $$

$$ \hat{U}_n = \frac{2}{n} - \frac{1}{2} \lambda $$

First order conditions

$$ \lambda = \frac{\hat{U}_g}{\hat{U}_n} \Rightarrow \lambda = \frac{g^*}{n^*} \Rightarrow \frac{1}{g} = \frac{4}{n} \Rightarrow n = 4g $$

Apply constraint $\Rightarrow$ 

$$ 3g + 2n = I \Rightarrow g^* = \frac{I}{5} $$

$$ n^* = \frac{4I}{5} $$
E) (2 pts) Suppose that she decides to increase her allocation to gasoline and natural gas by $10 a month. Approximately how much will her optimal utility increase?

\[ \frac{d}{dI} \hat{u}(I) = \lambda^* \]

\[ \lambda^* = \frac{1}{g^r} = \frac{5}{I} \]

\[ \Delta I = 10 \]

So \[ \Delta \hat{u} = \lambda^* \Delta I = \frac{50}{I} \]