Final Exam
15 March 2011

Instructions

- Please turn your OFF your cell phone, PDA, iPod, etc..
- There are 6 questions that will be scored out of 70 points. Problem 1 is worth 20 points and each other is worth 10 points, with 3 bonus points on problem 2. Thus 100%=70 points.
- Please write neatly and show all your work.
- You may bring in one page of notes, written in your own hand.
- Calculators may be used, but I have written the exam so that you generally don’t need to.
- Take your time, and check your answers.

Use the Method of Thinking!!! And have a good spring break.

NAME:

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Problem 1 Chops. Each Problem is Worth 2 points

A) The demand function for a certain item is

\[ p(q) = \frac{3}{q+1} \]

What is the marginal revenue function?

\[ r = \frac{3q}{q+1} \quad \frac{dq}{r} = \frac{3}{q+1} - \frac{3q}{(q+1)^2} = \frac{3}{(q+1)^2} \]

B) What is the marginal revenue from A) when \( q = 2 \)?

\[ r'(2) = \frac{3}{3^2} = \frac{1}{3} \]

(2) no partial credit [p.c.]

C) What is \( \int 4x \, dx \)?

\[ 2x^2 + C \]

(1) (1)

D) What is \( \int_0^1 4x \, dx \)?

\[ 2x^2 \bigg|_0^1 = 2 - 0 \]

(1) (1)

E) Suppose that the antiderivative of \( 4x \) takes the value 1 when \( x = 0 \). What is the constant of integration?

\( c = 0 \)

(2) no p.c.

F) What is \( \int \frac{-6}{(x+5)^2} \, dx \)?

\[ -6(x+5)^{-1} + C \]

(1) (1)
G) What is $\int (e^y - e^{-y}) \, dy$?

$$\int \left( \frac{e^y}{e} + e^{-y} \right) \, dy + C$$

H) A manufacturer’s production function is $P = 20^{0.7} k^{0.3}$ where $t$ is labor and $k$ is capital. What are the marginal productivity functions.

$$\frac{dP}{dt} = 14 \cdot t^{-3} \cdot k^{0.7}$$

$$\frac{dP}{dk} = 6 \cdot \ln(k)$$

I) In the first phase of adoption of a product, growth is exponential at the rate of 7% per year. If the initial number of products sold was 150, what is the general formula for the number of products sold after $t$ years?

$$A(t) = 150 \cdot e^{0.07t}$$

J) A product is adopted according to $\frac{dA}{dt} = 0.01 A \left( 1 - \frac{A}{10,000} \right)$ where $A$ is the number of units sold and $t$ is measured in months. Early on, when $A$ is small, the sales are growing exponentially at approximately what annual rate?

$$\frac{1}{12} \text{ months} \Rightarrow \frac{1}{12} \text{ yr} \Rightarrow 0.12 \text{ yr}^{-1}$$

K) What value of $A$ corresponds to the level of current sales with the maximum rate of sales?
Problem 2

A) (5 points) In class, we discussed the formula for the Taylor polynomial for \( f(x) \) around the point \( x_0 \):

\[
T[f(x|x_0)] = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n
\]

Show that when \( x \) is small \( e^x = 1 + x + \frac{x^2}{2} \).

\[
f(x) = e^x
\]

\[
f'(x) = e^x
\]

\[
f''(x) = e^x
\]

\[
f(x) = f(0) + f'(0) + \frac{f''(0)}{2} x^2 = 1 + x + \frac{1}{2} f''(0) x^2
\]

2

3

B) (5 points) Use the results from Part A) to evaluate \( e^{.1} \).

\[
e^{.1}
\]

\[
1 + (.1) + \frac{1}{2} (.1)^2
\]

\[
= 1 + .1 + \frac{1}{2} (.01)
\]

\[
= 1.105
\]

2
C) Bonus (3 pts): Do the same to show that when $x$ is small $\ln(1+x) = x - \frac{x^2}{2}$ and then evaluate $\ln(0.9)$.

\[
\begin{align*}
  f(x) &= \ln(1+x) \\
  f'(x) &= \frac{1}{1+x} \\
  f''(x) &= \frac{1}{(1+x)^2}
\end{align*}
\]

\[
\begin{align*}
  f''(x) &= \frac{1}{(1+x)^2} \\
  &= 0 + x - \frac{x^2}{2} \mid _{0}^{1} \\
  &= 0.9 = 1 + x \\
  \implies x &= -0.1 \\
  \Rightarrow \ln(0.9) &= -0.1 - \frac{1}{2} (0.1)^2 \\
  &= -0.105
\end{align*}
\]
Problem 3 A dairy produces two types of cheese, A and B, at average costs 50 cents and 60 cents per pound respectively. When the selling prices of A is $p_A$ cents per pound and the selling price of B is $p_B$ cents per pound, the demands for cheeses A and B are $q_A = 250(p_B - p_A)$ and $q_B = 32000 + 250(p_A - p_B)$. Find the selling prices that yield a relative maximum profit. Verify that the profit is a maximum at these prices.

\[ \Pi = (p_A - 50)(250)(p_B - p_A) + (p_B - 60)[32000 + 250(p_A - p_B)] \]

\[ \frac{\partial \Pi}{\partial p_A} = 250(p_B - p_A) + 250(p_A - 50)(-1) + 250(p_B - 60) \]

\[ = -500p_A + 500p_B + 10250 = 500(-p_A + p_B - 5) \]

\[ \frac{\partial^2 \Pi}{\partial p_A^2} = -500 \quad \frac{\partial^2 \Pi}{\partial p_A \partial p_B} = 0 \]

\[ \frac{\partial \Pi}{\partial p_B} = 250(p_A - 50) + 32000 + 250(p_A - p_B) - 500(p_B - 60) \]

\[ = 500p_A - 1000p_B + 32000 - 12500 + 30000 \]

\[ = 500p_A - 1000p_B + 49500 = 500(p_A - 2p_B + 99) \]

\[ \frac{\partial^2 \Pi}{\partial p_A \partial p_B} = 0 \Rightarrow \quad -p_A + p_B = 5 \]

\[ p_A - 2p_B = -99 \]

\[ p_B^* = 94 \text{ cents} \]

\[ p_A^* = 89 \text{ cents} \]

\[ \frac{\partial^2 \Pi}{\partial p_B^2} = -1000 \quad D = (-500)(-1000) - (500)^2 > 0 \]

\[ \Rightarrow \quad \text{max} \]
Problem 4
A (5 points) Solve the differential equation \( \frac{dy}{dx} = e^{2x} + 3 \) with the initial condition \( y(0) = 0 \)

\[
y = \frac{1}{2} e^{2x} + 3x + C \quad y(0) = 0 \Rightarrow \frac{1}{2} + C = 0 \]

\[
y = \frac{1}{2} e^{2x} + 3x - \frac{1}{2} \]

B (5 points) Solve the differential equation \( \frac{dy}{dx} = \frac{e^{1/x}}{x} \) with the initial condition \( y(1) = 5 \)

\[
y' = 1 + \frac{7}{x} \Rightarrow y = x + 7 \ln(x) + C \]

\[
y(1) = 5 \Rightarrow 5 = 1 + 7 \ln(1) + C \Rightarrow C = 4 \]

\[
y = x + 7 \ln(x) + 4 \]
Problem 5
A) (5 points) If the marginal revenue for a product is \( \frac{dR}{dq} = 100 - 1.5\sqrt{2q} \) determine the corresponding demand function.

\[
R' = 100 - 1.5\sqrt{2q^{1/2}} = 100 - \frac{3}{2} \sqrt{2q^{3/2}}
\]

\[
R = 100q - \frac{3}{2} \sqrt{2q^{3/2}} + C
\]

\[
= 100q - \sqrt{2q^{3/2}} + C \quad (2)
\]

\[
R(b) = 0 = C = 0 \quad (1)
\]

\[
R = 100q - \sqrt{2q^{3/2}} = 40 - 8 \quad (1)
\]

\[
p = \frac{R}{q} = 100 - \sqrt{2q^{1/2}} = 100 - \sqrt{2q} \quad (1)
\]
Problem 5 continued

B) (5 points) If the marginal cost of a product is \( \frac{dc}{dq} = q^2 + 7q + 6 \) and fixed costs are 2500, what is the total cost of producing six units?

\[ C = \frac{1}{3}q^3 + \frac{7}{2}q^2 + 6q + C_1 \]  \hspace{1cm} (2)

\( C(0) = 2500 \Rightarrow C(q) = \frac{q^3}{3} + \frac{7}{2}q^2 + 6q + 2500 \]  \hspace{1cm} (1)

\[ C(6) = \frac{6^3}{3} + \frac{7}{2} \times 36 + 36 + 2500 \]  \hspace{1cm} (1)

\[ = 72 + 126 + 36 + 2500 \]

\[ = 2734 \]  \hspace{1cm} (1)
Problem 6 The demand function for a product is \( p(q) = 0.01q^2 - 1.1q + 30 \) and the supply equation is \( p(q) = 0.01q^2 + 8 \). Determine the consumers' and producers' surpluses when the market equilibrium has been reached.

\[
\begin{align*}
P &= 0.01q^2 - 1.1q + 30 \\
\Rightarrow q_0 &= 20 \\
p_0 &= (0.01)(20)^2 + 8 = 12
\end{align*}
\]

\[
C_5 = \int_{0}^{20} (0.01q^2 - 1.1q + 30 - 12) \, dq = \left[ \frac{0.01q^3}{3} - \frac{1.1q^2}{2} + 18q \right]_0^{20}
\]

\[
= \left( \frac{0.01(20)^3}{3} - \frac{1.1(20)^2}{2} + 18(20) \right) - \left( \frac{0.01(0)^3}{3} - \frac{1.1(0)^2}{2} + 18(0) \right)
\]

\[
= \frac{800}{3} - 220 + 360 = 166.67
\]

\[
P_5 = \int_{0}^{20} \left[ 12 - 0.01q^2 - 8 \right] \, dq = \left[ 12q - \frac{0.01q^3}{3} \right]_0^{20}
\]

\[
= 240 - \frac{800}{3} = \frac{2}{3} \times 80 = 53.33 = 53\frac{1}{3}
\]