AMS 11B First Exam Answer Key

Note

- You can discuss questions about what you did wrong with the TAs or me, during section or office hours.

- You can argue for points only with me, during my office hours (the Thurs and Fri times on the web site) or by appointment.

MM
Problem 1. Find the indicated partial derivatives

A (2 points) For \( f(x, y) = \exp(3x + 2y^2) \) find \( f_x \) and \( f_y \).

\[
\frac{f_x}{f_y} = \exp[3x + 2y^2] \times 3x \; \text{and} \; \exp[3x + 2y^2] \times 4y
\]

(1 pt)

(1 pt)

B (3 points) For \( f(x, y) = \ln(3x^2 + 9y^3) \) find \( f_x, f_{xx}, \) and \( f_y \).

\[
\begin{align*}
\frac{f_x}{f_y} &= \frac{1}{3x^2 + 9y^3} \times 6x \\
\frac{f_{xx}}{f_{yy}} &= \frac{6}{3x^2 + 9y^3} - \frac{36x}{(3x^2 + 9y^3)^2} \\
\text{or} & \quad \frac{6(3x^2 + 9y^3) - 36x}{(3x^2 + 9y^3)^2}
\end{align*}
\]

(1 pt)

(1 pt)

C (5 points) For \( w(x, y, z) = e^{xy^3z} \) find \( w_x, w_y, w_z \) and \( w_{yz} \).

\[
\begin{align*}
\frac{w_x}{w_y} &= y^3ze^{xy^3z} \\
\frac{w_y}{w_z} &= x^3y^2ze^{xy^3z} \\
\frac{w_z}{w_x} &= xy^3ze^{xy^3z} \\
w_{yz} &= 3xy^2e^{xy^3z} + (xy^3)3y^2ze^{xy^3z}
\end{align*}
\]

(1 pt)

(1 pt)

Simplifying not needed
Problem 2
Captain Peter Blood has 1000 pounds sterling to invest in additional canons, $K$, that cost 80 pounds sterling each or in additional crew $L$, for which he has to pay a signing bonus of 20 pounds sterling. He knows that the loot provided by the new canons and crew is $Q(K, L) = 120 \cdot K \cdot L^{1/2}$.

A) (2 points) What is the relationship between the 1000 pounds, the number of new canons, and the number of new crew he can hire?

$$1000 = 80K + 20L$$

(1 pt) (1 pt)

B) (4 points) What is the marginal rate of change of loot with respect to crew when he has 20 canons and 25 crew members?

$$\frac{\partial Q}{\partial L} = 120 \cdot K \cdot \frac{1}{2} \cdot L^{-1/2}$$

(2 pts)

when $K = 20$  $L = 25$

$$\frac{\partial Q}{\partial L} = 120 \cdot 20 \cdot \frac{1}{2} \cdot \frac{1}{25}$$

(2 pts)

$$= 240$$  simplifying not needed

C) (4 points) A consumer eats either Wheaties or Cheerios, with milk, for breakfast but sometimes (if the cereal or milk is too expensive) will eat toast with jam. Let $q_w(p_W, p_C, p_M)$ denote her demand for Wheaties when the price of Wheaties, Cheerios and milk are $p_W, p_C, p_M$ respectively.

i) What is the sign of $\frac{\partial q_w}{\partial p_w}$ and why?  ii) What is the sign of $\frac{\partial q_w}{\partial p_C}$ and why?  iii) What is the sign of $\frac{\partial q_w}{\partial p_M}$ and why?

$$\frac{\partial q_w}{\partial p_w} < 0$$  when Wheaties are more expensive, she wants them less

(1 pt)

$$\frac{\partial q_w}{\partial p_C} > 0$$  when Cheerios are more expensive, she wants Wheaties more

(1 pt)

$$\frac{\partial q_w}{\partial p_M} < 0$$  when milk is more expensive, she eats toast and jam, less demand for Wheaties

(1 pt)
Problem 3
A (4 points) Suppose that \( g(u, v) = \ln(u^2 + 3v) \), and that \( u = r^2 + s, v = r - s \). What are \( \frac{\partial r}{\partial u} \) and \( \frac{\partial v}{\partial u} \)?

\[
\frac{\partial u}{\partial r} = \frac{2u}{u^2 + 3v} \quad \frac{\partial v}{\partial r} = \frac{3}{u^2 + 3v} \quad \frac{\partial u}{\partial s} = 2r \quad \frac{\partial v}{\partial s} = 1 \quad \frac{\partial s}{\partial r} = 1 \quad \frac{\partial s}{\partial s} = -1
\]

\[
g_r = \frac{\partial g}{\partial u} u_r + \frac{\partial g}{\partial v} v_r = \frac{2u}{u^2 + 3v} \cdot 2r + \frac{3}{u^2 + 3v} \cdot 1 = \frac{2(r^2 + s) + 3}{(r^2 + s)^2 + 3(r - s)}
\]

\[
g_s = \frac{\partial g}{\partial u} u_s + \frac{\partial g}{\partial v} v_s = \frac{2u \cdot 1 + 3(-1)}{u^2 + 3v} = \frac{2(r^2 + s) - 3}{(r^2 + s)^2 + 3(r - s)}
\]

B (6 points) The looting capacity, \( L(C, S) \) of one of Captain Blood's ships, depends upon the size of his crew \( C \), measured in persons, and on the number of sacks, \( S \) measured in cubic meters, they have for holding material according to \( L(C, S) = 100 \cdot \frac{C}{20 + C} \cdot \frac{S}{30 + S} \).

i) What are the units of 20 and 30 in the denominators? ii) What is the rate of variation of looting capacity vary with the number of sacks? iii) Suppose that the size of his crew depends upon treatment, \( t \), and food \( f \) according to \( C = 6t^{0.5} f^{0.3} \). What is the rate of variation of looting capacity with respect to food? with respect to treatment?

\[
\frac{\partial L}{\partial S} = \frac{30}{(30 + S)^2} \quad \frac{\partial L}{\partial C} = \frac{100 \cdot S}{30 + S} \cdot \frac{20}{(20 + C)^2}
\]

\[
\frac{dC}{dt} = 3 \cdot 2^{0.5} \cdot 1.8 \cdot f^{0.3} 
\]

\[
\frac{dS}{df} = \frac{100 \cdot 3}{30 + S} \cdot \frac{20}{(20 + C)^2} \quad \frac{dL}{df} = \frac{\partial L}{\partial C} \frac{dC}{df} + \frac{\partial L}{\partial S} \frac{dS}{df}
\]

\[
\frac{dL}{dt} = \frac{\partial L}{\partial C} \frac{dC}{dt} + \frac{\partial L}{\partial S} \frac{dS}{dt} = 100 \cdot \frac{S}{30 + S} \cdot \frac{20}{(20 + C)^2} \cdot 3 \cdot 2^{0.5} \cdot 1.8 \cdot f^{0.3}
\]
Problem 4  
A (4 points) Write the first three terms of the Taylor polynomial (ie $T_3[f(x)|x_0]$) for $f(x) = \frac{1}{x+1}$ around $x_0 = 0$.

\[
\begin{align*}
    f'(x) &= -\frac{1}{(x+1)^2} \\
    f''(x) &= \frac{2}{(x+1)^3} \\
    f'''(x) &= \frac{6}{(x+1)^4} \\
\end{align*}
\]  
(2 pts)

\[
T_3[f(x)|0] = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \frac{1}{6} f'''(0)x^3 \\
= 1 - x + x^2 - x^3  
\]  
(1 pt)

B (6 points) In fisheries economics today there is a commonly used tool called AD Model Builder, where AD stands for “Automatic Differentiation”. In this tool, one inputs a function and the computer takes the derivative, but the user still must understand how to apply the results. Thus, imagine a function $R(x,y)$ for which we know that $R(20, 30) = 100$, $R_x(20,30) = 5$, $R_y(20,30) = 3$, $R_{xx}(20,30) = 1$, $R_{xy}(20,30) = 2$, $R_{yy}(20,30) = 3$.

i) Write the second order Taylor polynomial, $T_2[R(x,y)|(20,30)]$. ii) Use it to evaluate $R(22,28)$.

\[
T_2[R(x,y)|(20,30)] = R(20,30) + R_x(20,30)(x-20) + R_y(20,30)(y-30) \\
+ \frac{1}{2} \left[ R_{xx}(20,30)(x-20)^2 + 2R_{xy}(20,30)(x-20)(y-30) + R_{yy}(20,30)(y-30)^2 \right] \\
= 100 + 5(x-20) + 3(y-30) \\
+ \frac{1}{2} \left[ 4(x-20)^2 + 4(x-20)(y-30) + 3(y-30)^2 \right].
\]  
(2 pts)

\[
R(22,28) = 100 + 5 \cdot 2 + 3 \cdot (-2) \\
+ \frac{1}{2} \left[ 4 + 4(2)(-2) + 3 (-2)^2 \right]  
\]  
(1 pt)