§ 14.1

The differential

\[ y = f(x) \]

Define

\[ dy = f'(x) \Delta x \quad \text{< this is the differential} \]

And

\[ f(x + \Delta x) = f(x) + dy \]

\[ = f(x) + f'(x) \Delta x \]

\[ \text{First order Taylor polynomial} \]

\[ y = f(x) \]

\[ 1 \]

\[ 1 \]

\[ \Delta x \]

\[ x \]
Why is this helpful

\[
\max \ U(x,y) \\
given \ g(x,y) = I
\]

\[
\hat{U}(x,y,\lambda|I) = U(x,y) - \lambda \left[ g(x,y) - I \right]
\]

follow the recipe

\[
\frac{\partial \hat{U}^*}{\partial I} = \lambda \quad \hat{U}^* = \hat{U}(x^*(I), y^*(I), \lambda^*(I)|I)
\]

The rate of change of the optimal value of utility given I

Economy gets better in —, and income goes from

\[ I \to I + \Delta I \]
How much does the utility change?

1. We could solve the problem.

2. Use the differential

\[ \Delta u = \frac{\Delta I}{\Delta I} = I^* \]

An approximation for how much the utility changes \( \Delta I \) for \( \Delta I \).

Another reason: \( \ln(1.05) = \ln(1 + .05) \)

Apply the differential

\[ f(x) = \ln(x) \]

\[ f'(x) = \frac{1}{x} \]

True value

\[ \ln(1.05) \approx 0.04879 \]

\[ x = 1 \quad \Delta x = .05 \]

The differential is \( f'(x) \Delta x = \frac{1}{1} \cdot .05 = .05 \)

\[ f(x + \Delta x) = \ln(1.05) = f(x) + f'(x) \Delta x \]

\[ = \ln(1) + .05 = .05 \]
It would be very cool if we could begin with a derivative, like
\[ \frac{\Delta u^*}{\Delta t} \]
and end with the function \( u^*(t) \).

Section 14.2 Antiderivatives and The Indefinite Integral

If
\[ F'(x) = f(x) \]
we say that \( F(x) \) is the antiderivative of \( f(x) \).

\[
\begin{array}{ccc}
\text{\( f(x) \)} & \text{\( F(x) \)} & \text{constant} \\
\hline
x & \frac{1}{2}x^2, \quad \frac{1}{2}x^2 + C & \\
x^3 & \frac{1}{4}x^4 + 16, \quad \frac{1}{4}x^4 + C & \\
x & \ln(x) + C & \\
\frac{1}{x} & -2x & \\
e^{-2x} & \frac{1}{2}e^{-2x} + C & -2x + C
\end{array}
\]
Some new symbology

\[ \int f(x) \, dx \] means find the antiderivative of \( f(x) \)

\[ \int \] integral sign

Integration is the process of finding the antiderivative

\[ \int f(x) \, dx \] is called the indefinite integral

When you see \( \int 3z^2 \, dz \), you think "This asks us to find a function whose derivative is \( 3z^2 \)"

\[ z^3 + C \]
Some general properties of integrals

1. \( \int f(x) \, dx = F(x) + C \)
   
   if and only if \( \frac{d}{dx} F(x) = f(x) \)

2. \( \frac{d}{dx} \left[ \int f(x) \, dx \right] = f(x) \)
   
   the derivative of the function whose derivative is \( f(x) \) is \( f(x) \)

3. \( \int \frac{d}{dx} [F(x)] \, dx = F(x) + C \)

find the function whose derivative is \( F(x) \) whose derivative will be \( \frac{d}{dx} F(x) \)

\[ \frac{d}{dx} [F(x) + C] = \frac{d}{dx} F(x) + \frac{d}{dx} C = \frac{d}{dx} F(x) \]
\[ \int k \, dx = kx + C \]

If \( a \neq -1 \),

\[ \int x^a \, dx = \frac{x^{a+1}}{a+1} + C \]

\[ \int \ln(x) \, dx = \ln(x) + C \]

\[ \int e^{kx} \, dx = \frac{1}{k} e^{kx} + C \]

\[ \int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx + C \]
\[ \int y(y+7) \, dy = \int [y^2 + 7y] \, dy \]

\[ = \int y^2 \, dy + \int 7y \, dy \]

\[ \int y(y+7) \, dy = \frac{1}{3} y^3 + \frac{7}{2} y^2 + C \]

Check: \( \frac{d}{dy} \left[ \frac{1}{3} y^3 + \frac{7}{2} y^2 + C \right] \)

\[ = y^2 + 7y + 0 = y(y+7) \checkmark \]

\[ \int \frac{(x-1)(x+3)}{5} \, dx = \frac{1}{5} \int (x-1)(x+3) \, dx \]

\[ = \frac{1}{5} \int \left[ x^2 + 3x - x - 3 \right] \, dx \]

\[ = \frac{1}{5} \int \left[ x^2 + 2x - 3 \right] \, dx \]

\[ = \frac{1}{5} \left[ \frac{1}{3} x^3 + x^2 - 3x \right] + C \]
\[ \int \frac{z^3 - 1}{z^2} \, dz = \int \left[ \frac{z^3}{z^2} - \frac{1}{z^2} \right] \, dz \]

\[ = \int \left[ z - z^{-2} \right] \, dz \]

\[ = \frac{1}{2} z^2 - (-z^{-1}) + C \]

\[ = \frac{1}{2} z^2 + z^{-1} + C \quad \text{okay.} \]

What do we need to find the constant

Do we know there is even a non-zero constant in the antiderivative

We need a specific value of the antiderivative?
Example

Find \( \int x^3 \, dx \) knowing that \( F(2) = 7 \)

\[
F(x) = \frac{1}{4} x^4 + C
\]

Since \( F(2) = 7 \) \( \Rightarrow \frac{1}{4} (2)^4 + C = 7 \)

\( \Rightarrow 4 + C = 7 \)

\( C = 3 \)

So

\[
F(x) = \frac{1}{4} x^4 + 3 \quad \text{v} \quad \int x^3 \, dx \text{ with value } 7 \text{ when } x = 2
\]