Historical Friday 28 JAN 1986

Challenger shuttle explosion

Deja vu all over again

- My job in AMS 11B: to teach calculus
- Your job in AMS 11B: to learn calculus

\[ e^{x^2 + 3y^2} \]

\[ \frac{S}{30+S} \]
Message from the Reader: What I am looking for when giving bonus points

1) Clean, clear answers (boxed helps) with clear train of thought

2) As simplified as possible (though won't take off points for simplification errors)

3) Neat layout, making it easy to read throughout

4) Completed work, all problems attempted thoroughly.

5) Ok to cross out cleanly. Preferable to erase. Make sure nothing that is boxed is crossed out (confusing).

Clean and clear presentation of work is an important habit

No more fighting about bonus points
Constrained Optimization

\[
\begin{align*}
&\text{maximize or minimize } f(x,y) \\
&\text{given a constraint } c(x,y) = c_0 \\
&\text{Create the augmented function } \\
&\hat{f}(x,y,\lambda) = f(x,y) + \lambda [c(x,y) - c_0] \\
\end{align*}
\]

"Eff-hat"  
"lambda"

and look for critical points of \( \hat{f}(x,y,\lambda) \)

To find the critical points, we set

\[
\begin{align*}
\frac{\partial \hat{f}}{\partial x} &= 0 \\
\frac{\partial \hat{f}}{\partial y} &= 0 \\
\frac{\partial \hat{f}}{\partial \lambda} &= 0
\end{align*}
\]
We obtain

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \\
\frac{\partial f}{\partial \lambda} &= 0 + (c(x,y) - c_0) = 0 \\
\end{align*}
\]

\[c(x,y) = c_0\]

Three equations to be solved for the critical point(s) in \((x,y,\lambda)\)
A factory produced 2 items with 200 units per week:

\[ q_1 + q_2 = 200 \]  \hspace{1cm} \text{[constraint]}

\[ \begin{align*}
\text{number of} & \quad \text{items of type 1} \\
\text{items} & \quad \text{type 2}
\end{align*} \]

Total cost is:

\[ TC(q_1, q_2) = 2q_1^2 + 6q_2 + q_2^2 + 200 \]

What production minimizes costs?

\[ \text{Step 1: Make the augmented function} \]

\[ ^{\wedge}TC(q_1, q_2, \lambda) = TC(q_1, q_2) + \lambda \left[ q_1 + q_2 - 200 \right] \]

\[ ^{\wedge}TC(q_1, q_2, \lambda) = 2q_1^2 + 6q_2 + q_2^2 + 200 + \lambda \left[ q_1 + q_2 - 200 \right] \]
Step 2: Find the derivatives

\[
\frac{\partial T_C}{\partial q_1} = 4q_1 + q_2 + \lambda \\
\frac{\partial T_C}{\partial q_2} = q_1 + 2q_2 + \lambda \\
\frac{\partial T_C}{\partial \lambda} = q_1 + q_2 - 200
\]

Step 3: Set derivatives equal to 0 and much about

\[
\begin{cases}
-\lambda = 4q_1 + q_2 \\
-\lambda = q_1 + 2q_2 \\
q_1 + q_2 = 200 \quad \text{[constraint]}
\end{cases}
\]

Both RHSs = \( \lambda \) means they are equal to each other

\[
4q_1 + q_2 = q_1 + 2q_2 \\
3q_1 = q_2 \quad \text{A relationship between } q_1 \text{ and } q_2 \text{ at the optimum}
\]
Apply the constant \( q_1 + q_2 = 200 \)

\[ q_1 + 3q_1 = 200 \]

\[ q_1 = 50 \]

\[ q_2 = 150 \]  

These give the cost minimizing values.

Step 4: How do we know that this is a minimizing value? We have a second derivative (not done here).
Least cost of production

\[ P = f(l, k) \]  
production function \[ l \sim \text{ labor} \]  
\[ k \sim \text{ capital} \]

\[ c(l, k) = p_l l + p_k k \]  
\( p_l \sim \text{ price of labor} \)  
\( p_k \sim \text{ price of capital} \)

How do we pick labor and capital to minimize costs given

\[ f(l, k) = P_t \]  
\( \text{target production} \)

\[ \text{constraint} \]

**Step 1 Augmented function**

\[ c(l, k, \lambda) = c(l, k) - \lambda \left[ f(l, k) - P_t \right] \]

\[ = p_l l + p_k k - \lambda \left[ f(l, k) - P_t \right] \]
Step 2: Take derivatives
\[
\frac{\partial}{\partial x} = \rho_e - \lambda \frac{\partial f}{\partial x}
\]
\[
\frac{\partial}{\partial k} = \rho_k - \lambda \frac{\partial f}{\partial k}
\]
\[
\frac{\partial}{\partial \lambda} = f(x,k) - \rho_k
\]

Step 3: Set the derivatives = 0 and work about
\[
\rho_e = \lambda \frac{\partial f}{\partial x}
\]
\[
\rho_k = \lambda \frac{\partial f}{\partial k}
\]
\[
f(x,k) = \rho_k \iff \text{constant}
\]
Solve the first two eqns for $\lambda$

$$\lambda = \frac{\phi e}{\phi_k}$$

$$\lambda = \frac{p_k}{\phi_k}$$

The two RHS must be equal

$$\frac{\phi e}{\phi_k} = \frac{p_k}{\phi_k}$$

$$\frac{\phi e}{\phi_k} = \frac{\phi_k}{\phi_k}$$

Interpretation: At the optimum, the ratio of marginal production with respect to labor to marginal production with respect to capital equals the ratio of their prices.

Ratio of prices = Ratio of marginal productivities
22. In a certain office, computers (c) and (d) are utilized for 8 c and 6 d hours, respectively. If daily output \( Q \) is a function of c and d, namely,

\[
Q = 18c + 20d - 2c^2 - 4d^2 - cd
\]

find the values of c and d that maximize Q.

To find the values of c and d giving a critical point for Q and then checking with the second derivative test:

**Step 1** Find the derivatives

\[
\frac{\partial Q}{\partial c} = 18 - 4c - d
\]

\[
\frac{\partial Q}{\partial d} = 20 - 8d - c
\]

\[
\frac{\partial^2 Q}{\partial c^2} = -4 \quad \frac{\partial^2 Q}{\partial d^2} = -8 \quad \frac{\partial^2 Q}{\partial c \partial d} = -1
\]
Step 2: At the critical point

\[
\frac{\partial Q}{\partial c} = \frac{\partial Q}{\partial d} = 0
\]

18 = 4c + d

20 = c + 8d

Solve top eqn for \( d \) \( \Rightarrow \) \( d = 18 - 4c \)

Use in the second eqn

20 = c + 8(18 - 4c)

Finish solving for \( c \) and \( d \)

Step 3: Second derivative test: we compute

\[
\left( \frac{\partial^2 Q}{\partial c^2} \right) \left( \frac{\partial^2 Q}{\partial d^2} \right) - \left( \frac{\partial^2 Q}{\partial c \partial d} \right)^2
\]

\[
= (-4)(-8) - (-1)
\]

\[
= 32 + 1 = 33 > 0 \quad \text{and} \quad \frac{\partial^2 Q}{\partial c^2} < 0
\]