4 Nov 09

Suppose \( f(z) \) and \( p(z) \) are functions

**Definition**

\[
\frac{d}{dz} r(z) = \lim_{h \to 0} \frac{r(z+h) - r(z)}{h}
\]

**Interpretations**

- \( r'(z) \) is slope of the tangent line
- \( r'(z) \) is the instantaneous rate of change

**Rules**

\[
\frac{d}{dz} \left[ r(z)p(z) \right] = r(z)p'(z) + r'(z)p(z)
\]

\[
\frac{d}{dz} \left[ \frac{r(z)}{p(z)} \right] = \frac{r'(z)p(z) - r(z)p'(z)}{p(z)^2}
\]
\[ \frac{d}{dz} \left[ r(z) + p(z) \right] = r'(z) + p'(z) \]

\[ \frac{d}{dz} \left[ e^{r(z)} \right] = e^{r(z)} \cdot r'(z) \]

\[ \frac{d}{dz} \left[ r(p(z)) \right] = \frac{dr}{dp} \cdot \frac{dp}{dz} \]

\[ \frac{d}{dz} \left[ \ln(p(z)) \right] = \frac{1}{p(z)} \cdot \frac{dp}{dz} \]

\[ \frac{d}{dz} \left[ p(z)^n \right] = n \cdot p(z)^{n-1} \cdot \frac{dp}{dz} \]

\[ \frac{d}{dz} \left[ b^z \right] = b^z \ln b \]

\[ \frac{d}{dz} \left[ b^{r(z)} \right] = b^{r(z)} \ln b \cdot \frac{dr}{dz} = b^{r(z)} \ln b \cdot r'(z) \]
\[
\frac{da}{dq} \left[ e^{kq} \right] = ke^{kq} \\
\frac{d}{dx} \left[ e^x \right] = e^x \\
\frac{d}{dq} \left[ e^q \right] = e^q
\]

More about elasticity

\[ f(q) \]

Point elasticity of demand

\[
\lim_{h \to 0} \frac{\text{% change in quantity for } q \to q+h}{\text{% change in price}} = f(q) \cdot \frac{1}{g} = \eta
\]
What if we are given

\[ q(p) \]

\[ \eta = \lim_{h \to 0} \frac{100 \cdot q(p+h) - q(p)}{100 \cdot \frac{h}{p}} \]

\[ = \lim_{h \to 0} \frac{q(p+h) - q(p)}{h} \cdot \frac{p}{h} \]

\[ = \frac{p}{q} \cdot \lim_{h \to 0} \frac{q(p+h) - q(p)}{h} \]

\[ \eta = \frac{p}{q} \frac{dq}{dp} \]
Summary

If we are given \( p(q) \)

\[ \eta = \frac{q}{\frac{df}{dq}} \]

If we are given \( q(p) \)

\[ \eta = \frac{p}{\frac{dq}{dp}} \]

This is an example of a general phenomenon involving inverse functions.

\[ x = f(y) \quad y = f^{-1}(x) \]

\[ \frac{dx}{dx} \left[ f(y(x)) \right] = \frac{dy}{dx} \left[ f(y(x)) \right] \]

\[ 1 = \frac{df}{dy} \cdot \frac{dy}{dx} \]
\[
\frac{d}{dy} \left[ x \right] = \frac{d}{dy} \left[ f(y) \right]
\]

\[
\frac{dx}{dy} = \frac{df}{dy} \cdot \frac{1}{f'}
\]

Using this in \( \parallel \) above

\[
1 = \frac{dx}{dy} \cdot \frac{dy}{dx}
\]

so

\[
\frac{dy}{dx} = \left( \frac{dx}{dy} \right)^{-1}
\]
Implicit Differentiation [finding \( \frac{dy}{dx} \) without explicitly knowing \( y = f(x) \)]

\[ xy - y - 15x = 3 \]

\[ y = \frac{3 + 15x}{x-1} \]

\[ \frac{d}{dx} \left[ xy - y - 15x \right] = \frac{d}{dx} [3] \]

\[ y \cdot \frac{dy}{dx} + x \cdot \frac{dy}{dx} - \frac{dy}{dx} - 15 = 0 \]

\[ y + \frac{dy}{dx} (x-1) = 15 \]

\[ \frac{dy}{dx} \cdot (x-1) = 15 - y \]

\[ \frac{dy}{dx} = \frac{15 - y}{x-1} \]

**Full stop**
Find \( \frac{dy}{dx} \)

\[ y \ln x = x \cdot e^y \implies y \text{ is a function of } x \]

\[
\frac{d}{dx} \left[ y(x) \cdot \ln(x) \right] = \frac{d}{dx} \left[ x \cdot e^y(x) \right]
\]

\[
\frac{dy}{dx} \cdot \ln(x) + \frac{y(x)}{x} = 1 \cdot e^y + \frac{d}{dx} \left[ e^y(x) \right]
\]

\[
\frac{dy}{dx} \cdot \ln(x) + \frac{y}{x} = e^y + e^y \cdot \frac{dy}{dx}
\]

\[
\frac{dy}{dx} \left[ \ln(x) - e^y \right] = e^y - \frac{y}{x}
\]

\[
\frac{dy}{dx} = \frac{e^y - \frac{y}{x}}{\ln(x) - e^y}
\]

**Full stop**
Find \( \frac{dy}{dx} \) if

\[ xe^y + y = 13x \]

\[ y \text{ is a function of } x \]

\[ 1 - e^y + x \cdot \frac{d}{dx} [e^y] + \frac{dy}{dx} = 13 \]

\[ e^y + [xe^y] \frac{dy}{dx} = 13 \]

\[ [1 + xe^y] \frac{dy}{dx} = 13 - e^y \]

\[ \frac{dy}{dx} = \frac{13 - e^y}{[1 + xe^y]} \]
Optimization Problems

\[ \text{S} \Rightarrow \text{amount of fencing} \]

Make a rectangular garden plot with perimeter S [ft of fencing]

Q. What shape of plot maximizes the area?
The founding of Carthage by Queen Dido in 814 BCE.

Vergil Aenid (64 BCE)

http://maps.google.com/maps?client=safr&rls=en&q=carthage+tun...7e%20x%20y%20x%20y=geocode_result&ct=image&resnum=1&ved=0CAIQ8gEWAA
Dido's problem: make a rectangle of the largest area in which three sides sum up to a length of hide $H$ and the fourth side is the ocean.
2W + L = H

Area = L \cdot W

But also \quad L = H - 2W

\[ A - \text{Area} = (H - 2W)W = HW - 2W^2 \]

\[ \frac{dA}{dW} = 0 \]
\[ A(w) = H \cdot W - 2w^2 \]

\[
\frac{dA}{dw} = H \cdot \frac{d}{dw}(w) - 2 \frac{d}{dw}(w^2)
\]

\[
= H \cdot 1 - 2 \cdot 2W
\]

\[
= H - 4W
\]

We set \( \frac{dA}{dw} = 0 \)

\[ H - 4W = 0 \]

\[ W = \frac{H}{4} \]
To the skeptic: \( H = 24 \text{ km} \)

\[ \frac{H}{3} = 8 \]

Area = 64 km\(^2\)

\[ \frac{H}{4} = 6 \]

\[ \frac{H}{2} = 12 \]

Area = 72 km\(^2\)