Exam #3 on Monday Nov 23

- Through Ch 12 & Lectures to Nov 16
  - Calculators allowed (but not necessary)
    No graphing calculators used

- Study session Sunday Nov 22
  4-7 pm
  Oakes Learning Center

- No Section or OH on Nov 25 (Wed's)
  to be sure you get the exam back on Nov 30

Optimal priacy

\[
\begin{align*}
  f'(x^*) & = 0 \\
  f''(x^*) & < 0 \\
  f'(x^*) & = 0 \\
  f''(x^*) & > 0
\end{align*}
\]
§ 13.2 Absolute Maxima/Minima

**Extreme value theorem**

If a function is continuous on a closed interval \([a, b]\), then the function has both a maximum and a minimum on \([a, b]\).

[Diagram of a function with labeled points indicating maxima and minima]
To use the extreme value theorem:

1) Find \( f(a), f(b) \) [why we need a closed interval]

2) Find all critical points: values of \( x \) such that

\[
f'(x) = 0
\]

check \( f''(x) \) for max/min

and then evaluate

\( f(x) \) at those points

3) Compare those with \( f(a), f(b) \)
Example 3.3

Concavity

\[ f'(x) = 0 \]

\[ f''(x) < 0 \leftarrow f'(x) < 0 \]

We say a function \( f(x) \) is concave up on an interval \((a, b)\) if \( f'(x) \) is increasing \( [f''(x) > 0] \).

We say a function is concave down . . . if \( f'(x) \) is decreasing \( [f''(x) < 0] \).
If a function \( f(x) \) is continuous at \( x = a \) and changes concavity there, then we say \( x = a \) is a point of inflection and \( f''(a) = 0 \) [or does not exist].

\[ f(x) = \frac{1}{3} x^3 \]

Normal Gaussian distribution

\[ f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left[-\frac{x^2}{2\sigma^2}\right] \]

(Bell-shaped curve)

Concave up

Concave down
find the indicated derivative at a given point.

\[ y = x^2 e^x, \quad y''', \quad (1, e) \]

\[ y''' = \frac{d}{dx} \left( \frac{d^2 y}{dx^2} \right) \]

\[ \frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) \]

\[ y = x^2 e^x \]

\[ \frac{dy}{dx} = 2x e^x + x^2 e^x \]

\[ \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[ 2xe^x + x^2 e^x \right] \]

\[ = 2 \left( e^x + xe^x \right) + 2xe^x + x^2 e^x \]

\[ = e^x \left[ 2 + 2x + 2x + x^2 \right] \]

\[ = e^x \left[ 2 + 4x + x^2 \right] \]
\[
\frac{d^3y}{dx^3} = \frac{d}{dx} \left\{ e^x \left[ 2 + 4x + x^2 \right] \right\} \\
= e^x \left[ 2 + 4x + x^2 \right] + e^x \left[ 4 + 2x \right]
\]

Now we plug in \( x = 1 \) \ldots
Differentiate

\[ y = \frac{(x^2 + 1)^{\frac{3}{2}} (x^2 + 2)^{\frac{2}{3}}}{(2x^3 + 6x)^{\frac{2}{15}}} \]

\[ \frac{dy}{dx} = \frac{\frac{3}{2} (x^2 + 1)^{\frac{1}{2}} (x^2 + 2)^{\frac{2}{3}} \ln (x^2 + 2) + \frac{2}{3} (x^2 + 1)^{\frac{3}{2}} (x^2 + 2)^{-\frac{1}{3}}}{(2x^3 + 6x)^{\frac{4}{15}}} \]

Remember the chain rule!!

\[ \frac{dy}{dx} = \left[ \frac{1}{2} (x^2 + 1)^{\frac{1}{2}} - 2x (x^2 + 2)^{\frac{1}{3}} + (x + 1)^{\frac{1}{2}} \frac{1}{2} (x^2 + 2)^{-\frac{4}{3}} \right] (2x^3 + 6x)^{\frac{4}{15}} \]
\[(\frac{(2x^3 + 6x)^{\frac{4}{5}}}{3})\]

\[6x^2 + 6\]
24. Find an equation of the tangent line to the curve

\[ x^3 + xy + y^2 = -1 \]

We need \( \frac{dy}{dx} \), which we find by implicit differentiation

\[
\frac{d}{dx} \left[ x^3 + xy + y^2 \right] = \frac{d}{dx} [-1] = 0
\]

Remember \( y = y(x) \) here

\[
3x^2 + 1 \cdot y + x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0
\]

\[
(x + 2y) \frac{dy}{dx} = -(3x^2 + y)
\]

\[
\frac{dy}{dx} = \frac{-(3x^2 + y)}{(x + 2y)}
\]
At \( x = a \) the value of \( y \) is found from

\[ a^3 + ay + y^2 = -1 \]

We use the quadratic formula and let us denote the value of \( y \) from here by \( y_a \)

Point: \( (a, y_a) \)

Slope: \[-\frac{3a^2 + y_a}{a + 2y_a}\]

Equation of tangent line

\[ \frac{y - y_a}{x - a} = -\frac{3a^2 + y_a}{a + 2y_a} \]
\[ \frac{d}{dx} \left[ e^{kx} \right] = ke^{kx} \]

23. Find \( \frac{dy}{dx} \) by implicit differentiation.

\[ (1 + e^{3x})^2 = 3 + \ln(x+y) \]

\[ \frac{d}{dx} \left[ (1 + e^{3x})^2 \right] = \frac{d}{dx} \left[ 3 + \ln(x+y(x)) \right] \]

\[ 2(1 + e^{3x}) \frac{d}{dx} e^{3x} = 0 + \frac{1}{x+y} \cdot (1 + \frac{dy}{dx}) \]

Remember the Chain Rule!

\[ 2(1 + e^{3x}) 3e^{3x} \cdot (x+y) = 1 + \frac{dy}{dx} \]

\[ \frac{dy}{dx} = 6 \left( 1 + e^{3x} \right) e^{3x} (x+y) - 1 \]
Find point elasticity of demand and determine whether demand is elastic, inelastic, or unit elastic.

\[ p = 150 - e^{q/100}; \quad q = 100 \]

\[ \eta = \frac{q}{p} \cdot \frac{1}{\frac{dq}{dp}} \]

\[ \varphi(q) = 150 - e^{q/100} \]

\[ \frac{d\varphi}{dq} = 0 - \frac{1}{100} e^{q/100} \]

Now when \( q = 100 \),

\[ p(100) = 150 - e^{100/100} = 150 - e \approx 147 \]

\[ \frac{dp}{dq} = -\frac{1}{100} e^{100/100} = -\frac{e}{100} \approx -\frac{3}{100} \]

(Definition: You're economists, you must know it.)

Calculator free.
when \( q = 100 \)

\[ \eta = \frac{150-e}{100} \cdot \left( \frac{1}{100} \right) \]

\[ \eta = \frac{150-e}{100} \quad \frac{d\eta}{dq} \bigg|_{q=100} = 100 \]

\[ \eta = \frac{150-e}{-e} = -\left(\frac{150-e}{e}\right) \]

\[ \approx = \frac{147}{e} \quad \approx -50 \]

\[ |\eta| > 1 \quad \text{elastic} \]

\[ |\eta| = 1 \quad \text{unit elasticity} \]

\[ |\eta| < 1 \quad \text{inelastic} \]
31. \( y = e^{x+y} \)

\[
y(x) = e^{x+y(x)}
\]

\[
\frac{d}{dx}[y(x)] = \frac{d}{dx}[e^{x+y(x)}]
\]

\[
\frac{dy}{dx} = e^{x+y(x)} \cdot [1 + \frac{dy}{dx}]
\]

\[
\frac{dy}{dx} \left(1 - e^{x+y} \right) = e^{x+y}
\]

\[
\frac{dy}{dx} = \frac{e^{x+y}}{1 - e^{x+y}}
\]

\[
y'' = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left[ \frac{e^{x+y}}{1 - e^{x+y}} \right]
\]
\[ y'' = \frac{(e^{x+y})(1 + \frac{dy}{dx})(1 - e^{x+y}) - e^{x+y}(-1)\left[1 + \frac{dy}{dx}\right]}{(1 - e^{x+y})^2} \]