AMS10

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AMS10A

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Name ____________________________

AMS 10/10A Final Exam

Section Day (Mon/Wed/Fri)___________

AMS10 or AMS10A? _________________

Spring Quarter 2009

INSTRUCTIONS:

1. AMS10A: Answer questions 1, 2 and 3 ONLY. You have 1.5 hours for the exam.

2. AMS10: Answer all questions 1 to 6. You have all 3 hours for the exam if you need them.

3. No calculators allowed (you won’t need them!)

4. Write your answers and do all working on this exam script if possible.

5. Be very clear as to how you arrived at a result (then assigning partial credit, if necessary, is easier). Mysterious answers that appear out of nowhere will not necessarily get full marks.

6. Write clearly, neatly, and in an orderly fashion. This also helps the graders give you more credit.

7. Notice that not all questions are worth the same amount of points.
1. **Applications of linear systems** (15 points total)

    Plants photosynthesize by taking carbon dioxide \( CO_2 \) from the air and combining it with water \( H_2O \) from the ground to produce complex carbohydrates such as glucose \( C_6H_{12}O_6 \), with oxygen \( O_2 \) as a by-product. What are the allowed ratios of the amounts of the four constituents (\( CO_2 : H_2O : O_2 : C_6H_{12}O_6 \)) in this chemical reaction?

    ANSWER:
2. **Matrix spaces** (20 points total)

This question concerns the following matrix representing a linear system in the vector space \( \mathbb{R}^2 \):

\[
A = \begin{pmatrix}
1 & 2 & 0 & 4 \\
1 & 3 & 2 & 7 \\
0 & 1 & 3 & 5
\end{pmatrix}
\]

(a) (2 points) Without doing any calculations, are the columns of \( A \) linearly independent? Justify your answer.
ANSWER:

(b) (5 points) Find a basis for the column space of \( A \).
ANSWER:

(c) (5 points) Find a basis for the null space of \( A \).
ANSWER:
(d) (3 points) What are the co-ordinates of the vector \( \mathbf{b} = (1, 2, 3) \) in the column space of \( A \) under the basis you defined?

ANSWER:

(e) (2 points) If a transformation \( T : \mathbb{R}^4 \to \mathbb{R}^3 \) is defined by the matrix \( A \), then is the transformation one-to-one? Justify your answer.

ANSWER:

(f) Write out the MATLAB commands you would use (i) to input the matrix \( A \), (ii) a command that would help you figure out the answers to parts (a) and (e), and (iii) a command that would figure out the kernel of the transformation \( T \).

ANSWER:

i. (1 point)

ii. (1 point)

iii. (1 point)
3. **Determinants and inverses of matrices** (15 points total)

   (a) (6 points) Find the unique solution of the following linear system using Cramer’s Rule.

   
   \[
   \begin{align*}
   3x_1 - 2x_2 &= 6 \\
   -5x_1 + 4x_2 &= 8 \\
   \end{align*}
   \]

   \text{ANSWER:}

   (b) (6 points) Find the solution of the same system using the inverse of the matrix from the matrix form of the above equations.

   \text{ANSWER:}

   (c) (3 points) Write out all the MATLAB commands you would use to do part (b) above. (Don’t forget to input the matrix and vector that you would need, etc.)

   \text{ANSWER:}
4. Orthogonality (20 points total)

(a) (2 points) Calculate the norm of the real 3-vector \( \mathbf{u} = (2, 1, -2) \).
ANSWER:

(b) (3 points) Calculate the norm of the complex 2-vector \( \mathbf{u} = (2 + i, i) \).
ANSWER:

(c) (4 points) Consider the following vectors in \( \mathbb{R}^3 \):
\[
\mathbf{v}_1 = (1, 0, 1) \quad \mathbf{v}_2 = (0, 1, 0) \quad \mathbf{v}_3 = (1, 0, 0)
\]
Are these vectors linearly independent? Justify your answer.
ANSWER:

(d) (3 points) Are the vectors orthogonal? Justify your answer.
ANSWER:
(e) (4 points) If not, then, from the original vectors, construct a set of 3 vectors that are orthogonal. Hint: Using projection, “fix” the vectors that are not orthogonal.
ANSWER:

(f) (2 points) Is the matrix whose columns are your 3 orthogonal vectors $A = [v_1 \ v_2 \ v_3]$ an orthogonal matrix? Justify your answer.
ANSWER:

(g) (2 points) What is the MATLAB command for finding the least squares solution of $Ax = b$?
ANSWER:
5. **Eigenvalues and eigenvectors** (15 points total)

Consider the following matrix:

\[ A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \]

(a) (4 points) What are the eigenvalues of the matrix \( A \)?

**ANSWER:**

(b) (4 points) What are the associated eigenvectors of the matrix \( A \)?

**ANSWER:**
(c) (2 points) Write down a matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$

ANSWER:

(d) (3 points) From the above, write the matrix form of $A^n$ as a single matrix.

ANSWER:

(e) (2 points) Write out the full MATLAB commands that you would use to find the eigenvalues and eigenvectors of the matrix $A$?

ANSWER:
6. **Applications of eigenvalues and eigenvectors: Stochastic matrices** (15 points total)

Aardvarks can only have blue or green eyes. Research\(^1\) shows that if a father aardvark has blue eyes, then his son’s chances of having blue eyes are 60%. However, if the daddy aardvark has green eyes, his son’s chances of green eyes are 70%. After many, many years of evolution, what is the ratio of blue-eyed aardvarks to green-eyed aardvarks in the male aardvark population (ratio blue:green)?

ANSWER:

\(^1\)Not really. I’m making this up!