7.3.5 \[ H_0: p = 0.5 \] \[ H_a: p < 0.5 \]

\( p \): proportion of boys

\[
Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \]

\( \hat{p} \): value obtained from sample

\( p_0 \): value in the null hypothesis

\( p_0 = 0.5 \)

\( \hat{p} = \frac{15}{50} = 0.3 \)

\( n = 50 \)

\[
Z_{obs} = \frac{0.3 - 0.5}{\sqrt{\frac{0.5 \times 0.5}{50}}} = -2.828
\]

\( Z_{obs} \): associated with the observed data if the null hypothesis is true.

Is -2.828 unusual under the null or not?

If unusual \( \Rightarrow \) reject null

If not \( \Rightarrow \) fail to reject
Two procedures to decide if it is unusual (equivalent)

**P-value:**

\[
\frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim N(0,1)
\]

if the null is true

\[
p-value = \Pr(Z < Z_{obs}) = \Pr(Z < -2.828)
\]

\[
= 0.0023 < \alpha = 0.05
\]

\[\Rightarrow\text{Reject } H_0\]

Critical value:

\[Z_{crit} = -1.645\]

\[Z_{obs} = -2.828\]

Since \[Z_{obs} \leq Z_{crit}\] then the observed z-score falls in the rejection region

\[\Rightarrow\text{Reject } H_0\]
Tests for 1 population

\[ N(0,1) \]

- Discrete

\[ \chi^2 \]

- Cont \[ \chi^2_{n-1} \]

\[ t_{n-1} \]

- Means

\[ N(\mu, \sigma) \]

- Variance Known

- Variance Unknown

Proportions
**TESTING MEANS (UNKNOWN VARIANCE)**

- The type of tests we will be focusing on look like
  
  \[
  \begin{align*}
  H_0: & \quad \mu = \mu_0 \quad \text{vs.} \quad H_a: \quad \mu \neq \mu_0 \\
  H_0: & \quad \mu = \mu_0 \quad \text{vs.} \quad H_a: \quad \mu > \mu_0 \\
  H_0: & \quad \mu = \mu_0 \quad \text{vs.} \quad H_a: \quad \mu < \mu_0
  \end{align*}
  \]

- Requirements:
  1. The sample is a simple random sample.
  2. The data is continuous.
  3. The value of the population standard deviation \( \sigma \) is unknown.
  4. The population is normally distributed or \( n \geq 30 \).
TESTING MEANS (UNKNOWN VARIANCE)

\( n = \) sample size.
\( \mu = \) population mean.
\( \sigma = \) population standard deviation.

Test statistic
\[ t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim T_{n-1} \]

Remember
\[ s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}} \]
TESTING MEANS (UNKNOWN VARIANCE)

- Rejection regions

\[ \text{Ha: } \mu \neq \mu_0 \quad \text{Ha: } \mu > \mu_0 \quad \text{Ha: } \mu < \mu_0 \]
TESTING MEANS (UNKNOWN VARIANCE)

- Critical values:
  - Two tail: $t_{a/2}$ and $t_{1-a/2}$
  - One tail: $t_{\alpha}$ (left tail) or $t_{1-\alpha}$ (right tail)

- p-values:
  - Two tail: $\Pr(|T| > t_{\text{obs}}) = 1 - 2\Pr(T < -t_{\text{obs}})$.
  - One tail: Left: $\Pr(T < t_{\text{obs}})$ Right: $\Pr(T > t_{\text{obs}})$.

- In this case, you need to use the t table!!! (but if $n$ is large, the $t$ table gives you the same value as the normal table)
**QUESTION**

The body temperature for 12 people was collected and the data is showed below. Use a 0.05 significance level to test the common belief that the mean body temperature of healthy adults is equal to 98.6°F.

- There is no evidence that the mean temperature is equal to 98.6°F.
- There is no evidence that the mean temperature is different from 98.6°F.
- There is evidence that the mean temperature is different from 98.6°F.

98.0 97.5 98.6 98.8 98.0 98.5 98.6 99.4 98.4 98.7 98.6 97.6

\[ H_0: \mu = \mu_0 \quad \quad H_a: \mu \neq \mu_0 \]

\[ \mu_0 = 98.6 \]

\[ \bar{x} = 98.391 \]

\[ S^2 = 0.2862 \Rightarrow S = 0.535 \]

\[ n = 12 \]

\[ \bar{x} = \frac{98.0 + 97.5 + \ldots + 97.6}{12} \]

\[ S^2 = \frac{1}{n} \left( (98 - 98.391)^2 + \ldots + (97.6 - 98.391)^2 \right) \]
\[ T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \sqrt{n} \left( \frac{98.391 - 98.6}{0.535} \right) = -1.3485 \]

**Critical region**

\[ \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \sim t_n \]

\[ \alpha = 0.05 \]

\[ t_{\alpha/2} = -2.01 \quad \text{and} \quad t_{1 - \alpha/2} = 2.01 \]

\( \because \) because the \( t \) is symmetric

Since \(-1.3485 \in (-2.201, 2.201) \to \) Fail to reject

**P-value**

\[ P_r \left( |T| > 1 - \alpha \right) = P_r(T > 1.3485) + P_r(T < -1.3485) \]

\[ = 2 P_r(T > 1.3485) \]

Too hard to get from a table.
TESTING VARIANCES

- The type of tests we will be focusing on look like
  \[ H_0: \sigma = \sigma_0 \text{ vs. } Ha: \sigma \neq \sigma_0 \]
  \[ H_0: \sigma = \sigma_0 \text{ vs. } Ha: \sigma > \sigma_0 \]
  \[ H_0: \sigma = \sigma_0 \text{ vs. } Ha: \sigma < \sigma_0 \]

- Requirements:
  1. The sample is a simple random sample.
  2. The data is continuous.
  3. The value of the population standard deviation \( \sigma \) is unknown.
  4. The population is normally distributed (note that a large sample is not enough in this case).
TESTING VARIANCES

\( n \) = sample size.
\( \mu \) = population mean.
\( \sigma \) = population standard deviation.

Test statistic

\[
X = \frac{(n - 1)s^2}{\sigma^2} \sim \chi^2(n - 1)
\]
THE $\chi^2$ DISTRIBUTION WITH N DEGREES OF FREEDOM

- Takes only positive values.
- Not symmetric!!!
- $E(\chi^2(n)) = n$
- $Var(\chi^2(n)) = 2n$
Testing testing variances

- Rejection regions

Ha: $\sigma \neq \sigma_0$  Ha: $\sigma > \sigma_0$  Ha: $\sigma < \sigma_0$
TESTING VARIANCES

- Critical values:
  - Two tail: $\chi^2(\alpha/2)$ and $\chi^2(1-\alpha/2)$
  - One tail: $\chi^2(\alpha)$ (left tail) or $\chi^2(1-\alpha)$ (right tail)

- In this case, you need to use the $\chi^2$ table!!!! This table is not symmetric.
QUESTION

The body temperature for 12 people was collected and the data is showed below. Use a 0.1 significance level to test the common belief that the standard deviation of the body temperature of healthy adults is equal to 0.62°F.

a) There is no evidence that the standard deviation of the body temperature is equal to 0.62°F.

b) There is evidence that the standard deviation of the body temperature is equal to 0.62°F.

There is no evidence that the standard deviation of the body temperature is different from 0.62°F.

c) There is evidence that the standard deviation of the body temperature is different from 0.62°F.

98.0 97.5 98.6 98.8 98.0 98.5 98.6 99.4 98.4 98.7 98.6 97.6

\[ H_0 : \sigma = 0.62 \quad H_\alpha : \sigma \neq 0.62 \]

\[ \sigma^2 = (0.62)^2 = 0.3844 \]

\[ s^2 = 0.2862 \]

\[ n = 12 \]

\[ X_{obs} = \frac{11 \times 0.2862}{0.3844} = 8.189 \]
Conclusion: Since 8.189 is in the control region we fail to reject H. 
There is no evidence that the variance is different from 0.2862°F².
(Standard deviation: 0.62°F)
Tests for Two Proportions

- The type of tests we will be focusing on look like
  - $H_0: p_1 = p_2$ vs. $H_a: p_1 \neq p_2$
  - $H_0: p_1 = p_2$ vs. $H_a: p_1 > p_2$
  - $H_0: p_1 = p_2$ vs. $H_a: p_1 < p_2$

- Requirements:
  1. The samples are simple random samples.
  2. The conditions for a binomial distribution are satisfied for both samples.
  3. The conditions for a normal approximation to the binomial have to be satisfied for both samples.
TESTING VARIANCES

\( n_1 = \) sample size for group 1.
\( n_2 = \) sample size for group 2.
\( p_1 = \) true proportion of successes in group 1.
\( p_2 = \) true proportion of successes in group 2.
\( x_1 = \) number of successes in group 1.
\( x_2 = \) number of successes in group 2.

\[
Z = \frac{(\hat{p}_2 - \hat{p}_1) - (p_2 - p_1)}{\sqrt{\hat{p}(1-\hat{p}) \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}} \sim N(0,1)
\]

\( \bar{p} = \frac{x_1 + x_2}{n_1 + n_2} \)
**Example**

Among 1070 children given a vaccine, 14 developed a flu. Among 532 children given a placebo, 95 developed a flu. Use a 0.05 significance level to test the claim that the proportion of flu cases among the vaccinated children is less than the proportion of flu cases among children given a placebo.

a) There is no evidence that the vaccine works.
b) There is evidence that the vaccine works.
c) There is no evidence that the vaccine does not work.
d) There is evidence that the vaccine does not work.

\[ H_0: P_1 = P_2 \quad \quad \quad \quad H_a: P_1 < P_2 \]

\[ P_1 = \text{proportion of infected kids that received vaccine} \]
\[ P_2 = \text{proportion of infected kids that did not get vaccine} \]

\[ \hat{p}_1 = \frac{14}{1070} = 0.0129 \quad \quad x_1 = 14 \quad n_1 = 1070 \]

\[ \hat{p}_2 = \frac{95}{532} = 0.1785 \quad \quad x_2 = 95 \quad n_2 = 532 \]

\[ \bar{p} = \frac{14 + 95}{1070 + 532} = 0.0680 \]
$z = \frac{(0.1785 - 0.0130) - 0}{\sqrt{0.068 \times 0.932 \left( \frac{1}{1020} + \frac{1}{632} \right)}}$

\[ = 12.39. \]

\[ z \sim N(0, 1) \]

\[ z_{\alpha/2} = 1.96 \]

Reject $H_0 \Rightarrow$ There is evidence that the vaccine has an effect on preventing flu.
TESTS FOR TWO MEANS, INDEPENDENT SAMPLES, EQUAL VARIANCES

○ The type of tests we will be focusing on look like
  H0: $\mu_1 = \mu_2$ vs. Ha: $\mu_1 \neq \mu_2$
  H0: $\mu_1 = \mu_2$ vs. Ha: $\mu_1 > \mu_2$
  H0: $\mu_1 = \mu_2$ vs. Ha: $\mu_1 < \mu_2$

○ Requirements:
  1. The samples are simple random samples.
  2. The data is continuous.
  3. Both samples are large (more than 30 observations) or both populations are normal.
  4. *The variance of both groups are approximately equal.*
Tests for Two Means, Independent Samples, Equal Variances

\[ n_1 = \text{sample size for group 1.} \]
\[ n_2 = \text{sample size for group 2.} \]
\[ \mu_1 = \text{True mean of group 1.} \]
\[ \mu_2 = \text{True mean of group 2.} \]

\[ T = \frac{(\bar{x}_2 - \bar{x}_1) - (\mu_2 - \mu_1)}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t(n_1 + n_2 - 2) \]

\[ \bar{s} = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} \]
Tests for Two Means, Independent Samples, Equal Variances

- The type of tests we will be focusing on look like:
  - $H_0: \mu_1 = \mu_2$ vs. $H_1: \mu_1 \neq \mu_2$
  - $H_0: \mu_1 = \mu_2$ vs. $H_1: \mu_1 > \mu_2$
  - $H_0: \mu_1 = \mu_2$ vs. $H_1: \mu_1 < \mu_2$

- Requirements:
  1. The samples are simple random samples.
  2. The data is continuous.
  3. Both samples are large (more than 30 observations) or both populations are normal.
  4. The variance of both groups is different.
Tests for Two Means, Independent Samples, Unequal Variances

\( n_1 = \) sample size for group 1.
\( n_2 = \) sample size for group 2.
\( \mu_1 = \) True mean of group 1.
\( \mu_2 = \) True mean of group 2.

\[ T = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_2 - \mu_1)}{\sqrt{(s_1^2 + s_2^2) / (n_1 + n_2)}} \sim t(u) \]

\[ \nu = \frac{(s_1^2 / n_1)^2 / (n_1 - 1) + (s_2^2 / n_2)^2 / (n_2 - 1)}{(s_1^2 / n_1)^2 / n_1 + (s_2^2 / n_2)^2 / n_2} \]
**QUIZ QUESTION**

- When investigating a relationship between birth weight and IQ researchers found that 258 subjects with extremely low birth weights had IQ scores at age 8 with a mean 95.5 and a standard deviation of 16.0. For 220 subjects with normal birth weights, the mean at age 8 is 104.9 and the standard deviation is 14.1. Does IQ score appear to be affected by birth weight.
  a) There is evidence that the IQ score is affected by birth weight.
  b) There is no evidence that the IQ score is affected by birth weight.
  c) There is evidence that IQ score is not affected by birth weight.
  d) There is no evidence that the IQ score is not affected by birth weight.

\[ H_0: \mu_1 = \mu_2 \quad H_a: \mu_1 \neq \mu_2 \]

- \( \mu_1 \) = normal weigh (mean IQ)
- \( \mu_2 \) = low birth weigh (mean IQ)

\[ \alpha = 0.05 \]

\( \hat{\sigma}_1 = 14.1 \) and \( \hat{\sigma}_2 = 16.0 \)

Since \( \hat{\sigma}_1 = 14.1 \) and \( \hat{\sigma}_2 = 16.0 \) it might be reasonable to assume \( \sigma_1 = \sigma_2 \)
Then

\[ T = \left( \frac{(\bar{X}_2 - \bar{X}_1) - 0}{\sqrt{\frac{s^2}{n_1} + \frac{s^2}{n_2}}} \right) \sim t_{(n_1 + n_2 - 2)} \]

\[ \bar{X}_1 = 104.9 \quad s_1 = 14.1 \]
\[ \bar{X}_2 = 95.5 \quad s_2 = 16.0 \]
\[ n_1 = 220 \quad n_2 = 258 \]

\[ s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = 229.68 \]

\[ \Rightarrow s = 15.15 \]

\[ T = \frac{(95.5 - 104.9)}{15.15 \sqrt{\frac{1}{220} + \frac{1}{258}}} = -6.76 \]

![Graph showing the t-distribution with the rejection of the null hypothesis]
TESTS FOR MATCHED PAIRS

- The type of tests we will be focusing on look like
  
  $H_0: \mu_1 - \mu_2 = 0 \quad \text{vs.} \quad H_a: \mu_1 - \mu_2 \neq 0$
  
  $H_0: \mu_1 - \mu_2 = 0 \quad \text{vs.} \quad H_a: \mu_1 - \mu_2 > 0$
  
  $H_0: \mu_1 - \mu_2 = 0 \quad \text{vs.} \quad H_a: \mu_1 - \mu_2 < 0$

- Requirements:
  
  1. The sample consists of matched pairs
  2. The samples are simple random samples.
  3. The data is continuous.
  4. The number of matched pairs is large (at least 30).
EXAMPLES OF EXPERIMENTS WITH MATCHED PAIRS

- When conducting an experiment to test the effectiveness of a low-fat diet, the weight of each subject is measured once before the diet and once after the diet.
- In a study to understand the genetic component of intelligence, researchers measure IQ on identical twins.
- In a test of the effects of a fertilizer on heights of trees, sample trees are planted in pairs, with one tree given the fertilizer treatment while the other tree is not given the treatment.
TESTS FOR MATCHED PAIRS

\[ n = \text{number of matched pairs.} \]
\[ d_i = \text{difference between the values in pair } i. \]
\[ \mu_1 = \text{True mean of group 1.} \]
\[ \mu_2 = \text{True mean of group 2.} \]
\[ s_d = \text{standard deviation of the differences.} \]
\[ \bar{d} = \text{average difference between pairs.} \]

\[ T = \frac{\bar{d} - (\mu_2 - \mu_1)}{\sqrt{s_d^2 / n}} \sim t(n - 1) \]
Two populations

- Proportions
- Means
  - Equal variance
  - Unequal variance
- Variances
  - Paired