probability

relative frequencies

with + without replacement
  - conditional probability

multiplication rule

independent events

mutually exclusive events

addition rule

working backwards
Properties.

impossible event: never happens.
0% of the time
probability 0.

if an event always happens, 100% of the time
probability 1.

chance between 0 and 100%
0 1.

chance that an event happens or that it
doesn't happen is 100%

chance of an event is 100 - chance of the
opposite event.

if an event has chance p of happening
opposite event has chance 1-p.
Example:

Box contains red and blue marbles.

draw one marble at random.

blue $\rightarrow $ $1$
red $\rightarrow $ $0$

Box 1: 3 red 2 blue
Box 2: 30 red 20 blue.

How often do you get red

Box 1: 3 out of 5
Box 2: 3 out of 5.

Ratio $\frac{\# \text{ red}}{\text{total } \#}$ $\Rightarrow$ $\# \text{ outcomes that you are interested in}$

total $\# \text{ outcomes}$.

$\uparrow$

when each outcome is equally likely.
list all possible outcomes.

Box 1:

\[
\begin{array}{c}
\text{R} \\
\text{R} \\
\text{R} \\
\text{B} \\
\text{B}
\end{array}
\]

count \( n \) that fit the condition = 3. divide by total number = \( \frac{3}{5} \)

With replacement.

Draw one, note colour, put it back. On subsequent draws, the contents are the same.

Without replacement.

If the first draw is \( \text{B} \), contents after 1st draw.

\( \text{R} \) chances of outcome on 2nd draw change.

\( \text{B} \) conditional probability.
Conditional Probability.

The occurrence of one event affects the chance of another event.

Deal 2 cards from shuffled deck. Deck.

Chance that 2nd card is ♠️? \( \frac{1}{52} \).

When I look at the 1st card (and it’s ♦️7) what’s the chance that 2nd card is ♠️? \( \frac{1}{51} \).

Conditional probability of the 2nd card being ♠️, conditioned on "given that the 1st card was not ♠️."
Multiplication Rule.

Draw 2 balls without replacement.

What are the chances of drawing:

1st: ⑨
2nd: ⑧

Approach #1.

Imagine 600 people drawing balls from a bag.

$\frac{1}{3}$ of them will get ⑨ first. 4 200 people

200 people ⑨ ⑩ how many draw ⑧ next.

$P( ⑨, ⑧ ) = \frac{4}{600} = \frac{1}{6}$
Approach #2.

\[ p(\text{R}) = \frac{1}{5}. \]

conditional on drawing \( \text{R} \), box contains

\[ \text{G R} \quad \leftarrow \text{prob of drawing G} = \frac{1}{2}. \]

\( \frac{1}{3} \) of the time, we succeed. \( \frac{1}{2} \) of the time.

so we succeed \( \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \) of the time.

The probability that two events will happen equals the probability that the first will happen times the probability that the second will happen given that the first has already happened.
What is the probability that the 1st card is a 7C, and the 2nd card is a Q?

\[ P(1\text{st card is 7C}) = \frac{1}{52}. \] (The 7C is equally likely to be in any of the 52 places.)

\[ P(2\text{nd card is Q given that 1st card is 7C}) = \frac{1}{51} \] (given that 1st card is 7C, the Q is equally likely to be in any of the remaining 51 places)

\[ P(1\text{st card is 7C and 2nd card is Q}) = \frac{1}{52} \times \frac{1}{51} = \frac{1}{2652}. \]
Example: chance of 1st 2 cards being aces?

\[ P(1\text{st in ace}) = \frac{4}{52} \]

\[ P(2\text{nd is ace given that 1st is ace}) = \frac{3}{51} \text{ - 3 aces left.} \]

\[ P(1\text{st 2 cards are both aces}) = \frac{4}{52} \times \frac{3}{51} \]

Example: toss coin twice.

What's chance of H T

\[ P(H \text{ on 1st throw}) = \frac{1}{2}. \]

\[ P(T \text{ on 2nd throw given that 1st throw was H}) = \frac{1}{2}. \]

this doesn't make any difference in this case.

\[ P(H T) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \]

prob. of 2nd event does not depend on the outcome of the first event.

- independent.
Two events are **independent** if the chances for the second event given the first are the same, regardless of the outcome of the first event.

otherwise they are **dependent**.

\[
\begin{array}{cccc}
1 & 1 & 2 & 2 \ 3 \\
\end{array}
\]  

two draws **with replacement**.

1st draw 1, what's chances of 12 on the 2nd draw?

by counting outcomes. \( p(12 \text{ on 2nd draw}) = \frac{2}{5} \),

because the contents of the box has not changed, as we are drawing **with replacement**.

\( \Rightarrow \) drawing with replacement gives **independent** events.
two draws without replacement.

After 1st draw: \(\square\).

\[ \square \ 2 \ 2 \ 3 \]

Contents have changed.

\[ p(\text{2 on 2nd draw given 1 on 1st draw}) = \frac{2}{4} = \frac{1}{2} \]

dependent events. - The probability for the 2nd event depends on the result of the 1st event.
Mathematical Notation.

Event $A$.

Probability of $A$ $p(A)$

Events $A$ and $B$

Conditional prob. of $A$ given $B$. $p(A | B)$

Multiplication Rule.

$p(A \text{ and } B) = p(A) \times p(B | A)$

$= p(B) \times p(A | B)$

Independence

$p(A | B) = p(A)$

$p(B | A) = p(B)$

For independent events. $p(A \text{ and } B) = p(A) \times p(B)$
A or B

either A happens, or B happens or both.

Start by looking at the case when it is impossible for both events to happen together.

- such events are called **mutually exclusive**

- the occurrence of one prevents the occurrence of the other.

Roll 2 dice, observe the sum

event A: sum is less than 3

event B: sum is greater than 6.

- A and B cannot happen together

- A and B are mutually exclusive.

For mutually exclusive events, the chance that at least one happens is the sum of the chance of each.
Example

Chance of top card being either H or S?

Approach 1: 13 H and 13 Spades.

\[
\Rightarrow \frac{26}{52} = \frac{1}{2}.
\]

Approach 2: chance of it being H = \(\frac{13}{52} = \frac{1}{4}\)

chance of it being Spade = \(\frac{13}{52} = \frac{1}{4}\)

\((\text{top card is H}) \text{ and } (\text{top card is Spade})\).

are mutually exclusive events.

\Rightarrow \text{ use addition rule.}

\[
\text{Chance of top card being either H or Spade} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]
Example.

What's the chance of at least one $\square$

when I roll 2 dice

$\Phi$ chance of $\square$ on white die $= \frac{1}{6}$.

$\Phi$ chance of $\square$ red $= \frac{1}{6}$.

while

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & x & x & x & x & x & x \\
2 & x & x & x & x & x & x \\
3 & x & x & x & x & x & x \\
4 & x & x & x & x & x & x \\
5 & x & x & x & x & x & x \\
6 & x & x & x & x & x & x \\
\end{array}
\]

why has the addition rule given the

wrong answer?

The two events (at least one $\square$) are not

(at least one $\square$ or any other)

exclusively

- both occur

when roll
Addition rule overcounts the outcome when both occur.

- If events are not mutually exclusive, addition rule gives a result that is too big.

to correct for the overcounting, subtract the chance that both happen.

\[
\frac{1}{6} + \frac{1}{6} - \frac{1}{36}.
\]

\(\square\) on red die \(\uparrow\) \(\square\) on white die

\[P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).\]

What about independent events?

can independent events be mutually exclusive?
Mutually exclusive events - knowing that one event has happened tells you a lot about the chance of the other event (in particular, the chance of the other event is zero).

- mutually exclusive - one, not the other, but not both.
- are not independent.

(contrast: independent events - prob. of one event does not change with the outcome of the other event).

independent event $\rightarrow$ multiply unconditional prob.

mutually exclusive $\rightarrow$ add probabilities.