Working Backwards.

Example: Roll a die 4 times
What are the chances of at least one four?

Think of the options

4 - - - 4 4 - - 4 4 4 -
- 4 - - 4 - 4 - - 4 -
- - 4 - - - 4 - 4 -
- - - 4 - - - - 4 -

We want to avoid listing all these combinations

⇒ look at the opposite event.

Chance of no fours in 4 rolls.

\[ \frac{5}{6} \text{ to the power of 4} \]

Then chance of at least one four

\[ 1 - \left( \frac{5}{6} \right)^4 \]
Thinking about these problems in terms of systems in \underline{series} or \underline{parallel}

\[ \begin{array}{c}
P_1 \\
\downarrow \quad \downarrow \\
P_2 & \text{system works if all components work,} \\
\downarrow \quad \downarrow \\
P_3
\end{array} \]

If components are \underline{independent} then

\[ \text{prob} \ P_1 \times P_2 \times P_3 \]

\[ \begin{array}{c}
P_1 \\
\downarrow \\
P_2 \\
\downarrow \\
P_3
\end{array} \quad \text{system works if any component works.} \]

\[ \rightarrow \text{system fails if all components fail} \]

\[ \text{prob fails} = (1-P_1) \times (1-P_2) \times (1-P_3). \]

\[ \text{prob. works} = 1 - (1-P_1)(1-P_2)(1-P_3). \]
Summary:

event $A$ \hspace{1cm} $0 \leq p(A) \leq 1$

opposite event \hspace{1cm} $p(\text{not } A) = 1 - p(A)$

addition \hspace{1cm} $p(A \text{ or } B) = p(A) + p(B) - p(A \text{ and } B)$

\hspace{1cm} $= 0$ if mutually exclusive

multiplication \hspace{1cm} $p(A \text{ and } B) = p(A | B) p(B)$

\hspace{1cm} $= p(B | A) p(A)$

\hspace{1cm} $= p(A) \times p(B)$

\hspace{1cm} if $A$ and $B$ are independent
Avoiding Counting.

Five draws with replacement

chance a) exactly two being red.

If 2 are red, 3 are green

one possibility \(RRGGG\)

with chance \(\frac{1}{10} \times \frac{1}{10} \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10}\)

Many other possibilities

\[RRGGG,\quad RGRGG,\quad RGGGR,\quad RGGGR,\quad GRRGG,\quad GGGRR,\quad GGGGR,\quad GGRRG,\quad GGGGR,\ldots\]

10 different ways in total
Each of the 10 has the same chance:

\[
\frac{1}{10} \times \frac{1}{10} \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10}
\]

The possibilities are mutually exclusive.

\( \Rightarrow \) Addition rule

Add up the prob. for the 10 options.

\[
10 \times \left( \frac{1}{10} \right)^{2} \times \left( \frac{9}{10} \right)^{3}
\]

The binomial coefficient tells us how many combinations there are.
The number of ways we can arrange \( n \) objects, where \( k \) are of one type, and \( (n-k) \) are of the other type.

\[
n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1
\]

\[
1! = 1
\]

\[
0! = 1 \quad (b.o.d)
\]

\[
2! = 2 \times 1 = 2
\]

\[
3! = 3 \times 2 \times 1 = 6
\]

\etc
Success has prob $p$

Failure $1 - p$

Probability of $k$ successes in $n$ trials

\[
\frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}
\]

If $n$ is fixed in advance

$p$ is same for all trials

Trials are independent

Example

Family with 4 children

What's chance of more girls than boys?

$\frac{3}{4}$ chance of 3 girls out of 4

\[
\begin{align*}
\frac{4!}{3! \ 1!} \ (\frac{1}{2})^3 \ (\frac{1}{2})^1 \\
+ \frac{4!}{4! \ 0!} \ (\frac{1}{2})^4 \times (\frac{1}{2})^0
\end{align*}
\]

\[= 0.25 + 0.06 = 0.31\]
Roll a die 6 times.

Chance of 4

\[ n = \text{# trials} = 6 \]

\[ p = \text{probability of success}. \]

\[ p(\text{rolling 3}) = \frac{1}{6} \]

\[ k = \text{# successes} = 4. \]

Chance of 4 in 6 rolls.

\[
\frac{6!}{4! \times (6-4)!} \times \left(\frac{1}{6}\right)^4 \times \left(1 - \frac{1}{6}\right)^{6-4}.
\]

\[
\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 2 \times 1} \times \left(\frac{1}{6}\right)^4 \times \left(\frac{5}{6}\right)^2
\]

\[
\geq 15 \times \left(\frac{1}{6}\right)^4 \times \left(\frac{5}{6}\right)^2
\]