2.2.7 Roots of quadratics

As we saw in the last lecture, some parabolas cross the $x$–axis, and some do not. Graphically, we know that there are 3 possible cases: a parabola can never cross the $x$–axis, it can just graze the $x$–axis at one point, or it crosses it at two different points:

Since a parabola is the graph of a quadratic function, this means that some quadratic functions $f(x) = ax^2 + bx + c$ have 2 roots, some have one root, and some do not. In other words,

- Sometimes there are 2 solutions to $f(x) = ax^2 + bx + c = 0$
- Sometimes there is only one solution to $f(x) = ax^2 + bx + c = 0$
- Sometimes there are no solutions to $f(x) = ax^2 + bx + c = 0$

Whether a quadratic has one or two roots is much more obvious from its factored form than from its expanded form.

**Examples**

- $f(x) = 3(x - 1)(x + 2)$
  
  $f(x) = 0$ implies $3 = 0$ or $(x - 1) = 0$ or $(x + 2) = 0$
  
  \[ \downarrow \text{impossible} \quad \downarrow x = 1 \quad \downarrow x = -2 \]
  
  $\Rightarrow$ 2 roots, $x = 1$ and $x = -2$

- $f(x) = \frac{1}{2}(x - 4)^2$
  
  \[ f(x) = 0 \Rightarrow \frac{1}{2} = 0 \text{ or } (x - 4) = 0 \]
  
  \[ \downarrow \text{impossible} \quad \downarrow x = 4 \]
  
  $\Rightarrow$ one root, $x = 4$

More generally:

- If $f(x)$ is in the factored form $f(x) = a(x - x_1)(x - x_2)$ then $f(x)$ has two roots, $x = x_1$, and $x = x_2$.

- If $x_1 = x_2$ ($x_1$ and $x_2$ are the same) then the factored form is $f(x) = a(x - x_1)^2(x - x_1) = a(x - x_1)^2$ and $x_1$ is the only root. It's called a "repeated" root because the term $x - x_1$ appears twice in the factored form.
2.2. QUADRATIC FUNCTIONS

In fact, there is a strict equivalence relationship between the two statements "f(x) can be factored" and "f(x) has roots": the first implies the second, and conversely, the second implies the first. This is often written mathematically as:

\[
\text{f(x) can be factored} \iff \text{f(x) has roots}
\]

The interesting thing about equivalence statements in logic is that if you have one, then you also have equivalence of the opposites:

\[
A \iff B \iff \text{not } A \iff \text{not } B
\]

For the case of factored forms and roots, the "opposite statements" then give

\[
f(x) \text{ cannot be factored} \iff f(x) \text{ does not have roots}
\]

**Examples**

- \( f(x) = x^2 + 4 \implies \text{cannot be factored, so should not have roots}\)
  
- \( f(x) = x^2 + 2x + 4 = x^2 + 2x + 1 + 3 = (x + 1)^2 + 3 \implies \text{cannot be factored, and doesn't have roots} \)

2.2.8 Factoring quadratics

Based on what we just saw, it would be nice to have simple tricks to tell us when a quadratic has roots or not, and what they are. This would then automatically tell us when a quadratic can be factored, and what the factored form is.

As we saw in the previous section, there are a few types of quadratics that can very easily be factored:

- \( x^2 + 2ax + a^2 = (x+a)^2 \)
- \( x^2 - 2ax + a^2 = (x-a)^2 \)
- \( x^2 - a^2 = (x-a)(x+a) \)
Examples

- \( f(x) = x^2 - 2 \quad \rightarrow \quad \text{looks like } (x^2 - a^2) \text{ with } a^2 = 2 \)

\[
\L = (x - \sqrt{2})(x + \sqrt{2})
\]

- \( f(x) = 2x^2 - 3 \quad \rightarrow \quad \text{looks like } x^2 - a^2 \), because

\[
\L = 2 \left( x^2 - \frac{3}{2} \right) = 2 \left( x - \sqrt{\frac{3}{2}} \right) \left( x + \sqrt{\frac{3}{2}} \right)
\]

or simply write

\[
2x^2 - 3 = \left( \sqrt{2}x^2 - \sqrt{3} \right) \left( \sqrt{2}x + \sqrt{3} \right) = \left( \sqrt{2}x - \sqrt{3} \right) \left( \sqrt{2}x + \sqrt{3} \right)
\]

- \( f(x) = x^2 + 6x + 9 \quad \rightarrow \quad \text{looks like } x^2 - 2ax + a^2 \)

\[
\L = (x + 3)^2
\]

- \( f(x) = 2x^2 + 4\sqrt{3}x + 10 \quad \rightarrow \quad \text{looks like } x^2 - 2ax + a^2 \) because

\[
\L = 2 \left( x^2 + 2\sqrt{3}x + 5 \right) = 2 \left( x + \sqrt{3} \right)^2
\]

- \( f(x) = -x^2 + 10x - 25 \quad \rightarrow \quad \text{looks like } x^2 - 2ax + a^2 \) because

\[
\L = - \left( x^2 - 10x + 25 \right)
\]

\[
\L = - \left( x - 5 \right)^2
\]

On the other hand, not every quadratic is in one of these three "ideal forms". What can we do if it isn't? As it turns out, another nice trick exists in that case, and is called "The quadratic formula".

The Quadratic Formula. Given the quadratic \( ax^2 + bx + c \),

- Calculate the discriminant \( D = b^2 - 4ac \)

- If \( D < 0 \) there are no solutions to the equation \( ax^2 + bx + c = 0 \), and the quadratic cannot be factored.

- If \( D = 0 \) there is one solution to the equation \( ax^2 + bx + c = 0 \), \( x = -\frac{b}{2a} \) and the quadratic can be factored as

\[
ax^2 + bx + c = a \left( x + \frac{b}{2a} \right)^2
\]
2.2. QUADRATIC FUNCTIONS

- If $D > 0$ there are two solutions to the equation $ax^2 + bx + c = 0$, which are $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$, and the quadratic can be factored as $a(x - x_1)(x - x_2)$.

EXAMPLES:

- What are the solutions (if any) to the equation $f(x) = 2x^2 - 3x + 1 = 0$? What is the factored form of $f$?

  $D = (-3)^2 - 4(2)(1) = 9 - 8 = 1 \rightarrow 2$ solutions

  $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-3) \pm \sqrt{1}}{2(2)} = \frac{3 \pm 1}{4} = \left\{ \begin{array}{l} \frac{4}{4} = 1 \\ \frac{2}{4} = \frac{1}{2} \end{array} \right.$

  So $f(x) = 2(x - x_1)(x - x_2) = 2 \left( x - \frac{1}{2} \right)^2$

- What are the solutions (if any) to the equation $f(x) = x^2 + x - 6 = 0$? What is the factored form of $f$?

  $D = b^2 - 4ac = (1)^2 - 4(1)(-6) = 1 + 24 = 25 \rightarrow 2$ solutions

  $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm 5}{2} = \left\{ \begin{array}{l} -3 \\ 2 \end{array} \right.$

  So $f(x) = (x - x_1)(x - x_2) = (x - 3)(x + 2)$

- What are the solutions (if any) to the equation $f(x) = -2x^2 - 8x - 8 = 0$? What is the factored form of $f$?

  $D = b^2 - 4ac = (-8)^2 - 4(-2)(-8) = 64 - 64 = 0 \rightarrow 1$ root

  $x_0 = \frac{-b}{2a} = \frac{-(-8)}{2(-2)} = -2 \rightarrow$ no solutions

  So $f(x) = -2(x - x_0)^2 = -2(x + 2)^2$

- What are the solutions (if any) to the equation $f(x) = -x^2 + x - 6 = 0$? What is the factored form of $f$?

  $D = b^2 - 4ac = (1)^2 - 4(-1)(-6) = 1 - 24 = -23 \rightarrow$ no solutions

  Then $f(x)$ cannot be factored.
NOTE: In a few particular cases, this method can also help solve higher-order equations that can be reduced to a quadratic, as in these examples:

- What are the solutions (if any) to the equation \( f(x) = x^6 - 3x^3 - 9 = 0 \)?

  Suppose we write \( x = x^3 \), then \( x^6 - 3x^3 - 9 = \left( x^3 \right)^2 - 3x^3 - 9 = 0 \), a quadratic in \( x \)

  \[ D = (-3)^2 - 4(1)(-9) = 9 + 36 = 45 \quad \Rightarrow \text{two solutions} \]

  \[ x_{1,2} = \frac{-(-3) \pm \sqrt{45}}{2} = \frac{3 \pm \sqrt{45}}{2} \]

  So since \( x = x^3 \), \( x = \sqrt[3]{x} \) \[ x_1 = \sqrt{\frac{3 + \sqrt{45}}{2}} \quad x_2 = \sqrt{\frac{3 - \sqrt{45}}{2}} \]

- What are the solutions (if any) to the equation \( f(x) = x^4 - 2x^2 - 3 = 0 \)?

  If \( x = x^2 \), then \( x^4 - 2x^2 - 3 = 0 \)

  \[ D = (-2)^2 - 4(1)(-3) = 4 + 12 = 16 \]

  \[ x_{1,2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = \begin{cases} 6/2 = 3 \\ -2/2 = -1 \end{cases} \]

  So since \( x^2 = x \) we solve \( x = \sqrt{x} \)

  Case 1: \( x = 3 \) \[ x = \pm 3 \]
  Case 2: \( x = 1 \) \( x^2 = 1 \) has no solutions.

WHERE DOES THE QUADRATIC FORMULA COME FROM?

Understanding where the quadratic formula comes from is important because it makes use of neat mathematical tricks, and also gives you a way to "rediscover it" should you ever forget it.

**Step-by-Step:** From \( ax^2 + bx + c \)

- Factor \( a \):
  \[ a(x^2 + \frac{b}{a} x + \frac{c}{a}) \]

- Complete the square in bracket
  \[ x^2 + \frac{b}{a} x + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} \]

  \[ = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} \]

  with \( D = b^2 - 4ac \).
2.2. QUADRATIC FUNCTIONS

- If \( D < 0 \) then
  \[
  f(x) = a \left[ \left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2} \right]
  \]
  \[\text{root is minus a negative quantity is like adding a positive one}\]
  \[\Rightarrow f(x) \text{ cannot be factored.}\]

- If \( D = 0 \)
  \[
  f(x) = a \left( x + \frac{b}{2a} \right)^2
  \]
  so this is the factored form of \( f(x) \) and
  \[f(x) \text{ has one root, } x_0 = -\frac{b}{2a}\]

- If \( D > 0 \)
  \[
  f(x) = a \left[ \left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2} \right]
  \]
  \[\text{If } D \text{ is positive then this is like the difference of two squares}\]
  \[= a \left( x + \frac{b}{2a} - \frac{\sqrt{D}}{2a} \right) \left( x + \frac{b}{2a} + \frac{\sqrt{D}}{2a} \right)\]
  \[\Rightarrow \text{factored form}\]
  \[= a \left( x - \frac{b + \sqrt{D}}{2a} \right) \left( x - \frac{b - \sqrt{D}}{2a} \right)\]
  \[\Rightarrow \text{There are two solutions}\]
  \[x_1 = -\frac{b + \sqrt{D}}{2a}, \quad x_2 = -\frac{b - \sqrt{D}}{2a}\]

2.2.9 A fun application for quadratics

Quadratics were first studied seriously when it was realized that they universally describe the trajectory of thrown objects (c.f. Isaac Newton’s work). Let’s consider the following scenario...
If the white bird is thrown from 1m off the ground, at a velocity of 1m/s, and at a 45 degree angle from the horizontal, how far ahead will it land?

To answer this question, it may help to know that the trajectory of an object thrown at a velocity \( v_0 \) (in m/s), and angle \( \alpha \) from the horizontal, and from a height \( h_0 \), is given by

\[
h(x) = -\frac{10x^2}{2v_0^2 \cos^2 \alpha} + x \tan \alpha + h_0
\]

where \( x \) is the distance from launch position.

Here, from the text: \( h_0 = 1 \), \( v_0 = 1 \), \( \alpha = 45^\circ \)

\[
\Rightarrow \cos 45^\circ = \frac{\sqrt{2}}{2}
\]

\[
\Rightarrow \tan 45^\circ = 1
\]

so \( h(x) = -5x^2 + x + 1 = -10x^2 + x + 1 \)

To find where the bird lands, we need to solve \( h(x) = 0 \) (i.e., height = 0 when it hits the ground)

\[
D = b^2 - 4ac = (1)^2 - 4(-10)(1) = 1 + 40 = 41
\]

\[
x_{1,2} = \frac{-1 \pm \sqrt{41}}{-20} = \left\{ \begin{array}{l}
\frac{-1 + \sqrt{41}}{-20} = -0.29 \\
\frac{-1 - \sqrt{41}}{-20} = 0.37
\end{array} \right.
\]

The first root doesn't make sense (bird can't go back) but the second root is right: 0.37 m ahead.