2.2 Quadratic functions

Textbook Sections 2.3, 2.4 and 2.6

2.2.1 Definition and basic \( f(x) = x^2 \) function

The general expression for a quadratic function is
\[
f(x) = ax^2 + bx + c \quad \text{where } a, b, c \text{ are real numbers.}
\]
Some quadratic functions (but not all of them!) can be factored, so that:
\[
f(x) = a(x-x_1)(x-x_2) \quad x_1, x_2 \text{ real numbers.}
\]
The simplest example of a quadratic function is the function \( f(x) = x^2 \):

In fact, the graph of all quadratic functions is a parabola. The exact shape and position of the parabola depends on the coefficients of the quadratic. Different cases can arise:

- The parabola can open up or down \( (a) \)

- The parabola may or may not have roots.
  It can have 1 or 2 if it does. \( (b) \)

- The parabola can have a tangent with positive or negative slope at \( x = 0 \) \( (c) \)
2.2. QUADRATIC FUNCTIONS

In the next few sections, we will get a better intuition for the relationship between the graph of a parabola and the mathematical expression of the corresponding function.

2.2.2 Behavior as $x \to \pm \infty$

Whether a parabola opens "up" or "down" can very easily be determined simply by inspection of the quadratic term $ax^2$ in the function.

Let's consider two examples of quadratic functions:

- $f(x) = 3x^2 - 2x - 1$
- $g(x) = -2x^2 + x + 1$

and complete the following tables:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$3x^2$</th>
<th>$-2x$</th>
<th>$-1$</th>
<th>$f(x) = 3x^2 - 2x - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1000</td>
<td>3,000,000</td>
<td>2000</td>
<td>1000</td>
<td>300,999</td>
</tr>
<tr>
<td>-100</td>
<td>300,000</td>
<td>20</td>
<td>1</td>
<td>30199</td>
</tr>
<tr>
<td>-10</td>
<td>30,000</td>
<td>2</td>
<td>1</td>
<td>319</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>3000</td>
<td>-20</td>
<td>-1</td>
<td>279</td>
</tr>
<tr>
<td>100</td>
<td>30,000</td>
<td>-200</td>
<td>-1</td>
<td>297999</td>
</tr>
<tr>
<td>1000</td>
<td>3,000,000</td>
<td>-2000</td>
<td>-1</td>
<td>299999999</td>
</tr>
</tbody>
</table>

We notice that:

- For large $x$, the functions $f(x)$ & $g(x)$ are $\pm ax^2$
- If $a > 0$ the parabola opens up; if $a < 0$ it opens down

This is in fact true of all quadratics!

- For large enough $x$, $f(x) = ax^2 + bx + c \approx ax^2$
- The graph of $f(x)$ looks like the graph of $ax^2$ for large $x$.
  - If $a > 0$, $f(x)$ opens up.
  - If $a < 0$, $f(x)$ opens down.
2.2.3 Behavior as $x \to 0$

What the parabola looks like near the $y$-axis (i.e. when $x$ is close to 0) can also very easily be determined simply by inspection of the quadratic function, but this time, of the $ax + c$ bit.

Let's complete the following tables for $f(x)$ and $g(x)$ again, but this time, for small $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$3x^2$</th>
<th>$-2x$</th>
<th>$-1$</th>
<th>$f(x) = 3x^2 - 2x - 1$</th>
<th>$-2x - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>+3</td>
<td>+2</td>
<td>-1</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>-0.1</td>
<td>+0.03</td>
<td>+0.2</td>
<td>.</td>
<td>-0.77</td>
<td>-0.8</td>
</tr>
<tr>
<td>-0.01</td>
<td>+0.0003</td>
<td>+0.02</td>
<td>.</td>
<td>-0.0997</td>
<td>-0.998</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0003</td>
<td>-0.02</td>
<td>-1</td>
<td>-1.0197</td>
<td>-1.02</td>
</tr>
<tr>
<td>0.1</td>
<td>0.03</td>
<td>-0.2</td>
<td>-1</td>
<td>-1.17</td>
<td>-1.2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>-5</td>
</tr>
</tbody>
</table>

We notice that:

- As $x \to 0$ the squared term rapidly becomes very small so $f(x)$ and $g(x) \approx bx + c$
- The graph of $f(x)$ and $g(x)$ looks like that of a straight line for small $x$, with a positive slope if $b > 0$
- The graph of $ax^2 + bx + c$ is very similar to that of $bx + c$ for small $x$.
- The line $y = bx + c$ is the tangent line to the parabola $y = ax^2 + bx + c$ at $x = 0$. 
2.2. QUADRATIC FUNCTIONS

2.2.4 Vertex of a parabola

**Definition:**

The **vertex of a parabola** is the location of its minimum (if \( a > 0 \)) or maximum (if \( a < 0 \))

As we shall see shortly, there are many examples of application where we are interested in finding out what the coordinates of the vertex are. As it turns out, it is quite easy to find them, using a method called **completing the square**. To see how it works, let us first note that, in some cases, it’s actually very easy to find the vertex without any mathematical manipulations at all, simply by remembering the basic rules of graphical transformations.

- \( f(x) = x^2 \):

  - vertex at \((0, 0)\)

- \( f(x) = x^2 - 1 \)

  - shift \( x^2 \) down by 1 \( \rightarrow \)
  - vertex at \((0, -1)\)

- \( f(x) = -3x^2 + 4 \)

  - shift \( x^2 \) up, \( \rightarrow \)
  - vertex at \((0, 4)\)

- \( f(x) = (x - 2)^2 \)

  - shift \( x^2 \) right by 2 \( \rightarrow \)
  - vertex at \((2, 0)\)

- \( f(x) = 1 + (x + 3)^2 \)

  - shift \( x^2 \) left by 3, up by 1 \( \rightarrow \)
  - vertex at \((-3, 1)\)

- \( f(x) = -5(x - 3)^2 - 1 \)

  - opens down
  - shift down by 1, right by 3
  - vertex at \((3, -1)\)
In all of these examples we see that if the quadratic function \( f(x) \) is in the vertex form
\[
 f(x) = a(x-x_v)^2 + y_v
\]
than:
the coordinates of the vertex are \((x_v, y_v)\)

2.2.5 Completing the square

"Completing the square" in a quadratic expression basically means putting it in vertex form, i.e., rewriting \( f(x) = ax^2 + bx + c \) as \( f(x) = a(x-x_v)^2 + y_v \).

Note how the \( a \) is the same in both expressions – do you see why?
\[
a(x-x_v)^2 + y_v = a \left( x^2 - 2x_v x + x_v^2 \right) + y_v
\]
\[
= \frac{ax^2 - 2ax_v x + a x^2 + y_v}{\text{same } a!}
\]
To complete the square in practice involves:

* matching \( ax^2 + bx \) to the first two terms of the expanded vertex form, to get \( x_v \)

* then figuring out what \( y_v \) has to be to match \( c \)

\[
ax^2 - 2ax_v x = ax^2 + bx \Rightarrow -2ax_v = b \Rightarrow\]
\[
\begin{align*}
  x_v &= \frac{-b}{2a} \\
  y_v &= c - ax_v^2
\end{align*}
\]

**Example 1** \( f(x) = x^2 - 2x + 3 \)
\[
 x_v = \frac{-b}{2a} = - \frac{(-2)}{2(1)} = 1 \\
 y_v = 3 - (1)(1) = 2 \Rightarrow f(x) = (x-1)^2 + 2
\]

**Example 2**: \( f(x) = 3x^2 - 3x - \frac{1}{4} \).
\[
 x_v = \frac{-b}{2a} = - \frac{(-3)}{2(3)} = \frac{1}{2} \\
 y_v = -\frac{1}{4} - 3 \left( \frac{1}{2} \right)^2 = -\frac{1}{4} - \frac{3}{4} = -1
\]
\[
\Rightarrow f(x) = 3 \left( x - \frac{1}{2} \right)^2 - 1
\]

**Example 3**: \( f(x) = -\frac{1}{2}x^2 - 3x + 1 \).
\[
 x_v = \frac{-b}{2a} = - \frac{(-3)}{2 \left( -\frac{1}{2} \right)} = -3
\]
\[
 y_v = 1 - \left( -\frac{1}{2} \right)^2(-3)^2 = 1 + \frac{9}{4} = \frac{13}{4}
\]
\[
\Rightarrow f(x) = -\frac{1}{2} \left( x + 3 \right)^2 + \frac{13}{4}
\]

**Note**: There are other techniques for completing the square! See section /HW for examples.
Another method for completing the square also exists, that does not require memorizing the two formulas for the \( x \)– and \( y \)–positions of the vertex. They do, on the other hand, require that you recognize some of the standard formulas for expanding quadratics, as in

- \((x + s)^2 = x^2 + 2xs + s^2\)
- \((x - s)^2 = x^2 - 2xs + s^2\)

So, if we have the beginning of the expressions on the right, we can create the "perfect squares" on the left!

**EXAMPLE 1:** \( f(x) = x^2 + 2x \).

\[
\begin{align*}
3x^2 + 2x & = x^2 + 2x + 1 - 1 = (x + 1)^2 - 1
\end{align*}
\]

Looks like the beginning of \( x^2 + 2x + 1 \).

**EXAMPLE 2:** \( f(x) = 2x^2 - 2x \).

\[
\begin{align*}
2x^2 - 2x & = 2(x^2 - x) + 0
\end{align*}
\]

Looks like the beginning of \( x^2 - 2ax + a^2 \) with \( a = \frac{1}{2} \).

**EXAMPLE 3:** \( f(x) = x^2 - 2x + 3 \).

\[
\begin{align*}
x^2 - 2x + 3 & = x^2 - 2x + 1 + 2 = (x - 1)^2 + 2
\end{align*}
\]

**EXAMPLE 4:** \( f(x) = 3x^2 - 3x - \frac{1}{4} \).

\[
\begin{align*}
3x^2 - 3x - \frac{1}{4} & = 3\left(x^2 - x\right) - \frac{1}{4} = 3\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) - \frac{1}{4}
\end{align*}
\]

Looks like the beginning of \( x^2 - 2ax + a^2 \) with \( a = \frac{1}{2} \).

**EXAMPLE 5:** \( f(x) = -\frac{1}{2}x^2 - 3x + 1 \).

\[
\begin{align*}
-\frac{1}{2}x^2 - 3x + 1 & = -\frac{1}{2}\left(x^2 + 6x\right) + 1
\end{align*}
\]

Looks like the beginning of \( x^2 + 6x + q - q \) with \( q = \frac{9}{2} \).
2.2. QUADRATIC FUNCTIONS

2.2.6 Application: optimization problems

Exercise 9 page 171. Also see Example 2 page 166 of textbook.

A farmer with 4000 meter of fencing wants to enclose a rectangular plot that is adjacent to a river, as in the graph below. What is the largest area he can fence off this way?

(1) Create mathematical variables to "name" quantities of interest:

- \( W \) = width of field
- \( L \) = length of field
- \( A \) = area of field

(2) Create mathematical equations that relate these quantities:

- \( A = LW \) \( \rightarrow \) definition of an area
- \( 4000 = 2W + L \) \( \rightarrow \) from the question

(3) Use these equations to eliminate some variables:

E.g. \( L = 4000 - 2W \) \( \longrightarrow \) \( W = \frac{4000 - L}{2} \)

\[ A = (4000 - 2W)W \]

\( \longrightarrow \) Now we have \( A(W) = 4000W - 4W^2 \)

\( = (4000 - 4W)W = 4(1000 - W)W \)

This looks like an inverted parabola.

\( \longrightarrow \) To find max, use vertex formula:

\[ x = \frac{-b}{2a} = \frac{-4000}{-4} = 500 \text{ m} \]

\[ y = \frac{4(500)^2}{1000000} = 1 \text{ m} \]