6.7.8 Applications of trigonometric formula: the beating phenomenon

Watch video on sound beating: https://www.youtube.com/watch?v=IYeV2Wq82fw

As we see in the video, two musical notes played at the same time but with slightly different tones interfere with one another and produce a phenomenon called beating. To understand beating, note that sound is actually a wave, i.e. an oscillation of the air between the instrument and our eardrum. Different sounds are simply oscillations that have different periods. The equation that describes one sound wave is

\[ s(t) = A \sin \left( \frac{2\pi}{p} t \right) \]

\[ A = \text{amplitude} \]

\[ p = \text{period} \]

Note that we often think of the properties of sound in terms of the frequency of the sound wave, rather than its period. The frequency and the period are related by

\[ p = \frac{2\pi}{f} \]

\[ s(t) = A \sin \left( ft \right) \]

\[ f \text{ is the frequency} \]

Low frequency sounds are low-pitched sounds. High frequency sounds are high-pitched sounds. When an instrument plays together two notes of different pitch, but similar amplitude then the equation for the sum of the two waves is simply:

\[ s_1(t) = A \sin \left( \frac{p_1}{L} t \right) \]

\[ s_2(t) = A \sin \left( \frac{p_2}{L} t \right) \]

\[ p_1 \neq \text{frequencies} \]

\[ s(t) = A \left[ \sin \left( ft \right) + \sin \left( f_2t \right) \right] \]

Suppose now, as suggested in the video, that the two waves have very similar frequencies (equivalently, very similar periods). For instance, let’s pick \( f_1 = 2 \) and \( f_2 = 2.2 \), and graph the resulting sum of two waves:

![Graph of sum of two waves]

Here \( s(t) = \sin (2t) + \sin (2.2t) \)

\[ \rightarrow \text{There is a modulation of the amplitude of oscillation!} \]

This shows that the sum of two notes of nearly the same frequency, and with the same amplitude, results in a modulation of the amplitude of the sound: this is the beating phenomenon!
To understand the phenomenon, remember the formula derived in the previous lecture relating the product of two sines to the sum of two sines: and sums of \( \sin \) and \( \cos \):

\[
\sin(a+b) = \sin a \cos b + \cos a \sin b \\
\cos(a-b) = \cos a \cos b + \sin a \sin b
\]

As it turns out, we can use this formula to prove another formula:

\[
2 \sin (a+b) \cos (a-b) = \sin(2a) + \sin(2b)
\]

Indeed:

\[
2 \sin(a+b) \cos(a-b) = 2 \left[ \sin a \cos b + \cos a \sin b \right] \left[ \cos a \cos b + \sin a \sin b \right]
\]

\[
= 2 \left[ \sin a \cos a \cos b \cos b + \sin a \sin b \sin a \cos b + \cos a \cos a \sin b + \cos a \sin a \sin b \right]
\]

\[
= 2 \left[ \cos^2 b \sin a \cos a + \sin^2 a \sin b + \cos^2 a \sin b + \cos a \sin a \sin^2 b \right]
\]

\[
= 2 \left[ \sin a \cos a \left( \cos^2 b + \sin^2 b \right) + \cos b \sin b \left( \cos^2 a + \sin^2 a \right) \right]
\]

\[
= 2 \sin a \cos a + 2 \sin b \cos b = \sin 2a + \sin 2b
\]

Now suppose \( 2a = f_1 t \) and \( 2b = f_2 t \)

Then \( a = \frac{f_1 t}{2} \) and \( b = \frac{f_2 t}{2} \)

\[
2 \sin \left( \frac{f_1 t + f_2 t}{2} \right) \cos \left( \frac{f_1 t - f_2 t}{2} \right) = \sin f_1 t + \sin f_2 t
\]

Using this equation we can then see that

\[
\sin f_1 t + \sin f_2 t = 2 \sin \left( \frac{f_1 + f_2}{2} t \right) \cos \left( \frac{f_1 - f_2}{2} t \right)
\]

This shows that the sum of the two waves is also equal to the product of two waves, one whose frequency is the average of the two original ones, and one whose frequency is half the difference between the two original ones. Now, if the two initial frequencies are nearly-identical, then the average frequency \( (f_1 + f_2)/2 \) is nearly the same as \( f_1 \) or \( f_2 \), while the frequency difference \( (f_2 - f_1)/2 \) is very small. This low frequency is called the beat frequency.

When a low frequency signal multiplies a high-frequency one, the resulting function looks like the high-frequency oscillation, but instead of having a constant amplitude, the amplitude varies in time according to the low-frequency signal. This is exactly what we are seeing here.
6.8 Solving trigonometric equations

Textbook Section 6.3

Trigonometric equations are equations that involve trigonometric functions, and that need to be solved for the unknown variable. The tricky thing about trigonometric equations is that sometimes they do not have solutions, but when they do have solutions, they often have infinitely many of them – and all of them need to be found. Let’s work by examples to see what may happen.

Example of equations that do not have solutions.

- $\cos(4x + 2) = 5.$
  
  The cosine function values between -1 and 1, so it can never be equal to 5.

- $\cos^2 a + \sin^2 a = 2.$
  
  $\cos^2 a + \sin^2 a = 1$ for all $a$ so this is the same as $1 = 2$, which has no solutions.

- $\cos^2 y + 2 \sin^2 y = 3.$
  
  $\cos^2 y + 2 \sin^2 y = \cos^2 y + \sin^2 y + \sin^2 y = 1 + \sin^2 y$
  
  $\Rightarrow 1 + \sin^2 y = 3 \Rightarrow \sin^2 y = 2$
  
  but $\sin^2 y$ values between 0 & 1 and can never be 2.

Examples of basic trigonometric equations that have infinitely many solutions

- Solve the equation $\cos(x) = \frac{1}{2}$
  
  $\Rightarrow$ A defines the angles $\frac{\pi}{3} + 2\pi n$
  
  B defines the angles $-\frac{\pi}{3} + 2\pi n$
  
  for any integer $n$
  
  $\Rightarrow$ solutions are
  
  $x \in \left\{ \frac{\pi}{3} + 2\pi n, -\frac{\pi}{3} + 2\pi n \right\}$.
• Solve the equation \( \sin(3x) = \frac{\sqrt{3}}{2} \)

A defines the angle \( \frac{\pi}{3} + 2\pi n \)

B defines the angle \( \frac{2\pi}{3} + 2\pi n \)

So

\( 3x \) is either

\[ 3x = \frac{\pi}{3} + 2\pi n \Rightarrow x = \frac{1}{3} \left( \frac{\pi}{3} + 2\pi n \right) = \frac{\pi}{9} + \frac{2\pi n}{3} \]

\[ 3x = \frac{2\pi}{3} + 2\pi n \Rightarrow x = \frac{1}{3} \left( \frac{2\pi}{3} + 2\pi n \right) = \frac{2\pi}{9} + \frac{2\pi n}{3} \]

So \( x \in \left\{ \frac{\pi}{9} + \frac{2\pi n}{3}, \frac{2\pi}{9} + \frac{2\pi n}{3} \right\} \)

• Solve the equation \( 3\cos(x) = 1 \)

Thus is the same as \( \cos(x) = \frac{1}{3} \)

To solve this, we can write \( x = \cos^{-1} \left( \frac{1}{3} \right) \) but that only returns the value of \( x \) between 0 and \( \pi \), since that's the range of \( \cos^{-1}(x) \) function.

\( \Rightarrow \) This picks out the point A.

But we also have the point B, as well as all the multiples of \( 2\pi \) angles.

\( \Rightarrow \) \( A: x = \cos^{-1} \left( \frac{1}{3} \right) + 2\pi n \)

\( B: x = -\cos^{-1} \left( \frac{1}{3} \right) + 2\pi n \) \( (B \) defines minus the angle of \( A) \)

\( \Rightarrow \) \( x \in \left\{ \cos^{-1} \left( \frac{1}{3} \right) + 2\pi n, \ -\cos^{-1} \left( \frac{1}{3} \right) + 2\pi n \right\} \).
6.8. SOLVING TRIGONOMETRIC EQUATIONS

- Solve the equation \( \cos^2(x) + \cos(x) - 2 = 0 \)

\[
\text{Let } u = \cos(x) \quad \text{then} \quad u^2 + u - 2 = 0
\]

\[
u^2 + u - 2 = (u + 2)(u - 1) \quad \text{so}
\]

\[
(u + 2)(u - 1) = 0 \quad \Rightarrow \quad u = -2 \quad \text{or} \quad u = 1
\]

\[
u = -2 : \quad \cos(x) = -2 \quad \Rightarrow \quad \text{no solution}
\]

\[
u = 1 : \quad \cos(x) = 1 \quad \Rightarrow \quad x = 0 + 2\pi n
\]

\[
\text{so} \quad x \in \{2\pi n\}
\]

EXAMPLES OF MORE ADVANCED TRIGONOMETRIC EQUATIONS THAT REQUIRE USING SOME TRIGONOMETRIC FORMULAS

- Solve the equation \( \sin(2x) = \cos(x) \)

Use the double-angle formula:

\[
8\sin(x) = 2\sin(x)\cos(x) = \cos(x)
\]

\[
\Rightarrow \quad 2\sin(x)\cos(x) - \cos(x) = 0
\]

\[
\Rightarrow \quad (2\sin(x) - 1)\cos(x) = 0 \quad \Rightarrow \quad \sin(x) = 0
\]

\[
\cos(x) = 0
\]

\[
\Rightarrow \quad x = \frac{\pi}{2} + 2\pi n \quad (A)
\]

\[
\Rightarrow \quad x = \frac{-\pi}{2} + 2\pi n \quad (B)
\]

\[
2\sin(x) - 1 = 0
\]

\[
\Rightarrow \quad \sin(x) = \frac{1}{2}
\]

\[
\Rightarrow \quad x \in \left\{ \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n \right\}
\]

\[
\Rightarrow \quad x \in \left\{ \frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n \right\}
\]

\[
\Rightarrow \quad x \in \left\{ \frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n \right\}
\]
• Solve the equation $3 \cos(a) + 3 = 2 \sin^2(a)$

  Use the Pythagorean identity: $\cos^2 a + \sin^2 a = 1$ so

  $\sin^2 a = 1 - \cos^2 a$

  $\Rightarrow 3 \cos a + 3 = 2 \left(1 - \cos^2 a\right) = 2 - 2 \cos^2 a$

  $\Rightarrow 2 \cos^2 a + 3 \cos a + 3 - 2 = 0$

  $\Rightarrow 2 \cos^2 a + 3 \cos a + 1 = 0$  
  (let $u = \cos a$).

  $\Rightarrow 2u^2 + 3u + 1 = 0$  
  $(2u + 1)(u + 1) = 0$

  $\Rightarrow u = -1$  or  $u = -\frac{1}{2}$

  $\cos a = -1$  
  $\cos a = -\frac{1}{2}$

  $\Rightarrow a \in \left\{ -\pi + 2\pi n \right\}$

  \begin{align*}
  a &= \frac{2\pi}{3} + 2\pi n \\
  b: a &= \frac{\pi}{3} + 2\pi n
  \end{align*}

  EXAMPLES OF TRIGONOMETRIC FUNCTIONS THAT CAN BE SOLVED GRAPHICALLY

• Solve the equation $\sin(a) = \cos(a)$

  This time, recall that $\sin(a)$ is the $y$-coordinate of the point $P$ and $\cos(a)$ is the $x$-coordinate. So effectively we're solving $y = x$ on the unit circle.

  $\Rightarrow A: \quad a = \frac{\pi}{4} + 2\pi n$

  $B: \quad a = \frac{3\pi}{4} + 2\pi n$
• Solve the equation \( \sin(a) = 2 \cos(a) \)

Let's apply the same method. This time

\[
\sin a = y, \quad \cos a = x \quad \Rightarrow \quad y = 2x
\]

This looks like the angle \( \frac{\pi}{3} \), but is it?

To be sure, note that any point on the unit circle also satisfies

\[
x^2 + y^2 = 1 \quad \Rightarrow \quad x^2 + (2x)^2 = 1 \quad \Rightarrow \quad 5x^2 = 1
\]

So \( x^2 = \frac{1}{5} \) \( \Rightarrow \) \( x = \frac{1}{\sqrt{5}} \)

So thus is not \( \frac{\pi}{3} \).

To find the angle \( a \), we now solve

\[
\cos a = x
\]

• If \( x = \sqrt{\frac{1}{5}} \) then \( \cos a = \sqrt{\frac{1}{5}} \) \( \Rightarrow \) \( a = \cos^{-1} \left( \sqrt{\frac{1}{5}} \right) + 2\pi n \)

• If \( x = -\sqrt{\frac{1}{5}} \) then \( \cos a = -\sqrt{\frac{1}{5}} \) \( \Rightarrow \) \( a = \cos^{-1} \left( -\sqrt{\frac{1}{5}} \right) + 2\pi n \)

\( \Rightarrow \) \( a \in \left\{ \cos^{-1} \left( \sqrt{\frac{1}{5}} + 2\pi n \right) , \cos^{-1} \left( -\frac{1}{\sqrt{5}} \right) + 2\pi n \right\} \).