Chapter 1

The notion of functions

Textbook Chapter 1

1.1 The concept of functions

Although the concept of a function was invented a very long time ago, it is very easy today to gain an intuitive notion of what functions are because of their natural role in most computer and/or web-based applications, in engineering, and in economics, etc.

Example 1A: Ordering diapers on Amazon

1 Giant diaper pack is $45.

When ordering \( n \) packs, the website returns the total cost as

\[
\text{cost} = \text{shipping} + (n \times 45) \times \text{tax rate}
\]

\[
= 10 + 4n \times 1.085 \quad \text{tax rate} = 8.5\%
\]

\[
\text{price}(n) = 10 + 48.325n
\]

Example 1B: Buying tomatoes at Safeway, self-checkout

Tomatoes are $4 per pound.

When using self-checkout, measured weight is \( w \) pounds \( \rightarrow \) the machine calculates price as

\[
\varphi(w) = 4 \times w \times \text{tax rate} = 4w \times 1.085 = 4.34w
\]

Example 2: Bank accounts. CD accounts offer various interest rates depending on the amount you put in: for instance, as of 09/19/15, at Chase, for a 2-year fixed term deposit you get interests of:

- 0.15\% per year for accounts under $10K

- 0.25\% per year for accounts between $10K and under $100K.
• 0.30% per year for accounts between $100K and under $250K.

The banker uses a computer function to tell you what your gain after 2 years will be as a function of your initial investment (cf. https://www.bankofinternet.com/calculators/apy-interest-calculator):

Example for under 10K: Start with \( x \) dollars, \( x < 10K \)

After one year: account has \( 1.0015x \)

After 2 years: account has \( 1.0015 \cdot 1.0015x \)

\[ \text{Gain is} \quad g(x) = (1.0015)^2 x - x = 0.003x \]

Similarly, for \( 10K < x < 100K \),

\[ \text{Gain is} \quad g(x) = (1.0025)^2 x - x = 0.005x \]

For \( 100K \leq x < 250K \):

\[ \text{Gain is} \quad g(x) = (1.003)^2 x - x = 0.006x \]

Example 3: Installing Cable TV (see Problem 22 page 105 of textbook).

Total cost = cost along road + cost off road

\[ = 500 \cdot \text{distance along road} + 700 \cdot \text{distance off road} \]

\[ = 500x + 700 \cdot L \]

To find \( L \), note that it’s a right-hand triangle, \( 8 \) \( \text{feet} \) the hypotenuse = \( L = \sqrt{2^2 + (5-x)^2} \)

So finally, cost \( C(x) = 500x + 700 \sqrt{4 + (5-x)^2} \)

1.2 Mathematical definitions

1.2.1 Definition of a function

Definition: A function.

Let \( A \) and \( B \) be two non-empty sets. A function from set \( A \) to set \( B \) is a rule that assigns to each element of \( A \), exactly one element of \( B \).

This definition is much more general than what we have seen so far, and covers a much wider class of functions, as for instance in the following examples:
1.2. MATHEMATICAL DEFINITIONS

Examples:

- Set $A =$ all people in this room
  Set $B =$ all integer numbers.

  Rule: to each person, the role associates that person’s age.

  Note: Every person has an age, but not every number has someone of that age!

- Set $A =$ all people in this room
  Set $B =$ all people in this room.

  Rule: to each person, the role associates their favorite person in the room.

  Note: This could be themselves!

Functions defined by mathematical expressions

In most of our work this quarter, however, we will only be considering functions that are defined as mathematical rules. These rules take one number, and associate to it another number. The mathematical rule is usually written as:

\[
\begin{align*}
\text{set of numbers} & \quad \rightarrow \quad \text{set of numbers} \\
X : A & \quad \rightarrow \quad B \\
X & \quad \rightarrow \quad f(X) = \quad \text{the mathematical expression}
\end{align*}
\]

Examples:

- A rule that returns \( n+1 \) for every integer \( n \):

  \[
  \begin{align*}
  \mathbb{N} & \rightarrow \mathbb{N} \\
  n & \rightarrow f(n) = n + 1
  \end{align*}
  \]

- A rule that returns \( x^2 \) for every real \( x \)

  \[
  \begin{align*}
  \mathbb{R} & \rightarrow \mathbb{R} \\
  x & \rightarrow f(x) = x^2
  \end{align*}
  \]

- A rule that returns \( -x \) for every negative \( x \), \( 0 \) if \( x = 0 \), and \( \sqrt{x} \) for every positive \( x \)

  \[
  \begin{align*}
  \mathbb{R} & \rightarrow \mathbb{R} \\
  x & \rightarrow \begin{cases} 
  f(x) = -x & \text{if } x \leq 0 \\
  f(x) = \sqrt{x} & \text{if } x > 0
  \end{cases}
  \end{align*}
  \]
1.2.2 Functions and their variables

As we saw earlier, mathematical functions are very useful tools to describe how things depend on one another. Usually, when trying to make a mathematical model of a real science or engineering problem, we are trying to understand "How does a quantity \( y \) depend on a quantity \( x \)?". This leads to the following definitions:

**Definitions:** Dependent and independent variables

- If a quantity \( y \) is related to a quantity \( x \) via \( y = f(x) \) then
  - \( x \) is the **independent variable**
  - \( y \) is the **dependent variable**

**Examples:**

- Price of Happiness \( n = \text{independent variable}, \ p = \text{dependent variable} \)
- Price of weight of tomatoes: \( w = \text{indep}, \ p = \text{dep} \)
- Cost of Comcast \( c(x) \): \( x = \text{indep}, \ c = \text{dep} \)

To evaluate a mathematical function, simply replace the independent variable in the mathematical formula by the number considered!

**Examples:**

- To find the cost of 10 diaper bags is: \( p(10) = 10 + 48.825 \times 10 = 498.25 \)
- My gains after a $200,000 investment are:
  - \( 100,000 \leq x \leq 200,000 \) then \( g(x) = 0.006x \)
  - \( g(200,000) = 0.006 \times 200,000 = 1200 \)
- The cost of putting the pole 1 mile away from the box is:
  \[
  c(1) = 500 \cdot 1 + 700 \sqrt{4 + (5-1)^2} = 500 + 700 \sqrt{20} \leq 3630.5
  \]

The function can also be applied to expressions instead of numbers. In that case, simply replace the dependent variable by the whole expression

**Examples:**

- If \( f(x) = 20x + 2 \) then
  \[
  f(x+3) = 20(x+3) + 2 = 20x + 62
  \]
- If \( f(x) = \sqrt{3x^2+2} \)
  then \( f(3x) = \sqrt{3 \cdot (3x)^2+2} = \sqrt{27x^2+2} \)
- If \( f(x) = x^2 \)
  \[
  f(x+h) = (x+h)^2 = x^2 + 2xh + h^2
  \]
  etc...
1.3 Domain of a function

**Domain of Definition of a Function:** The Domain of Definition of a function \( f \) consists of all of the values \( x \) for which we are allowed to or want to assign a value \( y = f(x) \).

- "allowed to" refers to the mathematical rules, i.e. when are you allowed to apply that rule to \( x \).
- "want to" refers to the physical problem considered, i.e. what are the values of \( x \) that make sense?

**Examples:**

1. \( f(x) = \frac{1}{x-1} \) Here \( x-1 \) cannot be 0 so \( x \) must be \( \neq 1 \)
   \[ \Theta = \mathbb{R} - \{1\} = (-\infty, 1) \cup (1, +\infty) \]

2. \( f(x) = \sqrt{x-1} \) Here \( x-1 \) cannot be negative so we must have \( x \geq 0 \) so \( x \geq 1 \)
   \[ \Theta = [1, +\infty) \]

3. \( f(x) = \frac{x-1}{x^2-1} \) Here we must both have: \( x^2-1 \neq 0 \)
   and \( 2-x \geq 0 \)

   The first implies \( x^2 \neq 1 \) so \( x \neq 1 \) and \( x \neq -1 \)
   The second implies \( 2 \geq x \) so \( x \leq 2 \)

   \[ \Theta = (-\infty, 2] - \mathbb{Q} \]

**Example 1:** What is the domain of definitions of the price of diapers? \( p(n) = 10 + 43 \cdot \text{.32}n \)

Mathematically, we could apply \( p \) to any number but it doesn't make sense to buy a "negative amount" of diapers, or a non-integer amount of diapers:

\[ \Theta = \text{all positive integers} = \mathbb{N} \]

**Example 2:** What is the domain of definitions of the cost of installing Cable? \( C(x) = 500 + 100 \sqrt{4 + (5-x)^2} \)

Mathematically, we could apply \( C \) to any number but it doesn't really make sense to have \( x \) negative or \( x \) greater than 5 (cost-wise)

\[ \Theta = [0, 5] \]
1.4 Operations on functions

Functions defined by mathematical expressions can be manipulated just like numbers: they can be added, subtracted, multiplied and divided by one another. In each case, the operation defines another function – the only tricky part being that one must sometimes re-evaluate the domain of definition.

Examples:

- If \( f \) and \( g \) are two functions of \( x \), then their sum is also a function of \( x \):
  
  * Given \( f(x) \) and \( g(x) \), we construct \( (f+g)(x) = f(x) + g(x) \)
  
  Example: \( c(x) \) is actually the sum of two functions of \( x \)
  
  \[
  c(x) = 500x + 700 \sqrt{4 + (5-x)^2}
  \]
  
  \[
  f(x) \quad g(x)
  \]

- If \( f \) and \( g \) are two functions of \( x \), then their difference is also a function of \( x \):
  
  * Given \( f(x) \) and \( g(x) \), we construct \( (f-g)(x) = f(x) - g(x) \)

- If \( f \) and \( g \) are two functions of \( x \), then their product is also a function of \( x \).
  
  * Given \( f(x) \) and \( g(x) \), we construct \( (f \cdot g)(x) = f(x)g(x) \)

  Example (see textbook problem 25 page 105).

  Volume of box = length of box \( \times \) area of base of box

  \[
  V(x) = h(x) \times A(x)
  \]

  \[
  h(x) = x \quad A(x) = (24-2x)^2
  \]

  \[
  V(x) = x(24-2x)^2
  \]

- If \( f \) and \( g \) are two functions of \( x \), then their quotient is also a function of \( x \):
  
  * Given \( f(x) \) and \( g(x) \), we construct \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \)

  Note that, for the case of the quotient, the domain of definition must now exclude all the values of \( x \) for which \( g(x) = 0 \).

  Example:

  \[
  f(x) = 3x+2 \quad g(x) = x-1 \quad \mathcal{D}_f = \mathbb{R}, \quad \mathcal{D}_g = \mathbb{R}
  \]

  \[
  \frac{f}{g}(x) = \frac{3x+2}{x-1} \quad \mathcal{D}_{\frac{f}{g}} = \mathbb{R} - \{1\}
  \]
1.5 Functions and their graphs

Graphs are an easy way to visualize how things depend on one another, and so they are an ideal visual way to represent a function.

**Definition: The Graph of a Function:**

The graph of a function $f(x)$ is the set of all the points with coordinates $(x, y)$ such that $y = f(x)$.

![Graph of a function]

The most basic graphing technique is

- to construct a table of values, with two columns: values of $x$ and corresponding values of $f(x)$
- draw the corresponding points with coordinates $(x, f(x))$.

The more points you have, the more accurate the representation of the function. In most cases, you can join the dots to complete the graph (but be careful of the few counterexamples).

**Example:**

$f(x) = x + 2$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

**Example:**

$f(x) = \sqrt{x+4}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
</tr>
</tbody>
</table>
CHAPTER 1. THE NOTION OF FUNCTIONS

SINGLE VALUE PROPERTY:

- Recall that saying "f is a function" only makes sense when there is a single value of y corresponding to each value of x. This means that while every function has a graph, not every graph can be the graph of a function!

- The Vertical Line Test: A graph corresponds to a function only if it passes the Vertical Line Test: if any vertical line on the graph intersects the line $y = f(x)$ more than once, then $f$ is not Single Valued, therefore $f$ is not a function.

EXAMPLES:

Some vocabulary: A lot more can be learned from a graph. You can immediately see whether

- a function is increasing or decreasing in any given interval
- a function has stationary points (minimum, maximum, or other)
- a function has roots (x-intercepts) or y-intercepts, or vertical asymptotes, or horizontal asymptotes
- a function is continuous or discontinuous, smooth or not smooth!

IMPORTANT NOTE: an empty circle on a graph means that the corresponding point has been "removed" from the graph at this position.
1.6 Techniques in graphing

As we saw, graphs of functions are very useful to get, "at a glance", some of the basic properties of the functions. However, graphing point by point be a little bit tedious, and there are a lot of faster techniques. They are based on knowing the graphs of "standard" functions, and how mathematical and geometrical manipulations of these graphs relate to one another.

1.6.1 The standard functions

The following standard functions are ones you should know how to graph, accurately, "by heart", from now on. As the course proceeds, we will continue adding to this list of functions.
1.6.2 Vertical translation

The graph of the function \( g(x) = f(x) + a \) can be obtained from that of the function \( f(x) \) by translating it vertically by an amount \( a \) (downward if \( a < 0 \) and upward if \( a > 0 \)).

Examples:

\[ \text{graph of } x^2 + 1 \]
\[ \text{graph of } x^2 \]
\[ \text{graph of } x^2 - 3 \]

1.6.3 Horizontal translation

The graph of the function \( g(x) = f(x - a) \) can be obtained from that of the function \( f(x) \) by translating it horizontally to the right by an amount \( a \) if \( a > 0 \) and to the left by an amount \( |a| \) if \( a < 0 \).

Examples:

\[ \text{graph of } \vert x \vert \]
\[ \text{graph of } \vert x + 5 \vert \]
\[ \text{graph of } \vert x - 3 \vert \]

Note: Horizontal translation can be confusing.

\[ f(x) \]
\[ f(x + a) \rightarrow \text{go towards} - \text{side} \]
\[ f(x - a) \rightarrow \text{go towards} + \text{side} \]

1.6.4 Reflections across the \( x \)- and \( y \)-axis

The graph of the function \( g(x) = -f(x) \) can be obtained from that of the function \( f(x) \) by reflection across the \( x \)-axis.
1.6. TECHNIQUES IN GRAPHING

The graph of the function \( g(x) = f(-x) \) can be obtained from that of the function \( f(x) \) by reflection across the \( y \)-axis.

1.6.5 Even and odd functions

Given that the graph of \( f(-x) \) can be obtained from that of the function \( f(x) \) by reflection across the \( y \)-axis (see above), we can deduce an important mathematical property of functions which have graphs that are symmetric with respect to the \( y \)-axis, as for instance:

**Definition:**
A function \( f(x) \) is called an even function if \( f(-x) = f(x) \). The graph of an even function is symmetric about the \( y \)-axis.

\[ f(x) = x^2 \]
\[ f(x) = (x+1)^2 = (x-1)^2 \]
\[ f(x) = \frac{1}{x} \]
\[ f(x) = -\frac{1}{x}. \]

\[ f(x) = x+1 \]
\[ f(x) = -x+1 \]

\[ f(x) = 1 \]

\[ f(x) = -1 \]

\[ f(x) = 0 \]

\[ f(x) = x \]

\[ f(x) = -x \]

\[ f(x) = 0 \]

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There is another interesting kind of symmetry, which is a point symmetry about the origin. The graphs of these functions look like this:

The functions whose graphs are point symmetric have the property $f(-x) = -f(x)$. Indeed:

\[
\begin{align*}
\quad f(-x) &= (-x)^3 = (-1)^3 \cdot x^3 = -x^3 = -f(x) \\
g(-x) &= \frac{1}{-x} = -\frac{1}{x} = -g(x).
\end{align*}
\]

**DEFINITION:** A function $f(x)$ is called an **odd function** if $f(-x) = -f(x)$. The graph of an odd function is point-symmetric about $(0, 0)$.

**EXAMPLES:** Are these functions odd or even or neither?

- $f(x) = 3x$
  \[f(-x) = 3(-x) = -3x = -f(x) \implies \text{ODD}\]
- $f(x) = -x + 1$
  \[f(-x) = -(-x) + 1 = x + 1 \n\neq f(x) \text{ or } -f(x) \implies \text{NEITHER}\]
- $f(x) = x^4$
  \[f(-x) = (-x)^4 = (-1)^4 \cdot x^4 = x^4 = f(x) \implies \text{EVEN}\]
- $f(x) = -x^5$
  \[f(-x) = -(-x)^5 = -(-1)^5 \cdot x^5 = x^5 = -f(x) \implies \text{ODD}\]
- $f(x) = \frac{2}{x}$
  \[f(-x) = \frac{2}{-x} = -\frac{2}{x} = -f(x) \implies \text{ODD}\]
- $f(x) = -\frac{1}{x^2}$
  \[f(-x) = -\frac{1}{(-x)^2} = -\frac{1}{x^2} = f(x) \implies \text{EVEN}\]