A.1 Significant Digits

Many of the numbers that we use in scientific work and in daily life are approximations. In some cases the approximations arise because the numbers are obtained through measurements or experiments. Consider, for example, the following statement from an astronomy textbook:

The diameter of the Moon is 3476 km.

We interpret this statement as meaning that the actual diameter $D$ is closer to 3476 km than it is to either 3475 km or 3477 km. In other words,

$$3475.5 \text{ km} \leq D \leq 3476.5 \text{ km}$$

The interval $[3475.5, 3476.5]$ in this example provides information about the accuracy of the measurement. Another way to indicate accuracy in an approximation is by specifying the number of significant digits it contains. The measurement 3476 km has four significant digits. In general, the number of significant digits in a given number is found as follows.

**Significant Digits**

The number of significant digits in a given number is determined by counting the digits from left to right, beginning with the leftmost nonzero digit.

<table>
<thead>
<tr>
<th>Number</th>
<th>Number of Significant Digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.43</td>
<td>3</td>
</tr>
<tr>
<td>0.52</td>
<td>2</td>
</tr>
<tr>
<td>0.05</td>
<td>1</td>
</tr>
<tr>
<td>4837</td>
<td>4</td>
</tr>
<tr>
<td>4837.0</td>
<td>5</td>
</tr>
</tbody>
</table>

Numbers obtained through measurements are not the only source of approximations in scientific work. For example, to five significant digits we have the following approximation for the irrational number $\pi$:

$$\pi \approx 3.1416$$

This statement tells us that $\pi$ is closer to 3.1416 than it is to either 3.1415 or 3.1417. In other words,

$$3.14155 \leq \pi \leq 3.14165$$

Table 1 provides some additional examples of the ideas we’ve introduced.
There is an ambiguity involving zero that can arise in counting significant digits. Suppose that someone measures the width $w$ of a rectangle and reports the result as 30 cm. How many significant digits are there? If the value 30 cm was obtained by measuring to the nearest 10 cm, then only the digit 3 is significant, and we can conclude only that the width $w$ lies in the range $25 \text{ cm} \leq w \leq 35 \text{ cm}$. On the other hand, if the 30 cm was obtained by measuring to the nearest 1 cm, then both the digits 3 and 0 are significant, and we have $29.5 \text{ cm} \leq w \leq 30.5 \text{ cm}$.

By using scientific notation, we can avoid the type of ambiguity discussed in the previous paragraph. A number written in the form

$$ b \times 10^n $$

where $1 \leq b < 10$ and $n$ is an integer

is said to be expressed in scientific notation. For the example in the previous paragraph, then, we would write

$$ w = 3 \times 10^1 \text{ cm} \quad \text{if the measurement is to the nearest 10 cm} $$

and

$$ w = 3.0 \times 10^1 \text{ cm} \quad \text{if the measurement is to the nearest 1 cm} $$

As the figures in Table 2 indicate, for a number $b \times 10^n$ in scientific notation the number of significant digits is just the number of digits in $b$. (This is one of the advantages in using scientific notation; the number of significant digits, and hence the accuracy of the measurement, is readily apparent.)

**TABLE 1**

<table>
<thead>
<tr>
<th>Number</th>
<th>Number of significant digits</th>
<th>Range of measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>37</td>
<td>2</td>
<td>[36.5, 37.5]</td>
</tr>
<tr>
<td>37.0</td>
<td>3</td>
<td>[36.95, 37.05]</td>
</tr>
<tr>
<td>268.1</td>
<td>4</td>
<td>[268.05, 268.15]</td>
</tr>
<tr>
<td>1.036</td>
<td>4</td>
<td>[1.0355, 1.0365]</td>
</tr>
<tr>
<td>0.036</td>
<td>2</td>
<td>[0.0355, 0.0365]</td>
</tr>
</tbody>
</table>

**TABLE 2**

<table>
<thead>
<tr>
<th>Measurement</th>
<th>Number of significant digits</th>
<th>Range of measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of the Earth:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$6 \times 10^{27} \text{ g}$</td>
<td>1</td>
<td>$[5.5 \times 10^{27} \text{ g}, 6.5 \times 10^{27} \text{ g}]$</td>
</tr>
<tr>
<td>$6.0 \times 10^{27} \text{ g}$</td>
<td>2</td>
<td>$[5.95 \times 10^{27} \text{ g}, 6.05 \times 10^{27} \text{ g}]$</td>
</tr>
<tr>
<td>$5.974 \times 10^{27} \text{ g}$</td>
<td>4</td>
<td>$[5.9735 \times 10^{27} \text{ g}, 5.9745 \times 10^{27} \text{ g}]$</td>
</tr>
<tr>
<td>Mass of a proton:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1.67 \times 10^{-24} \text{ g}$</td>
<td>3</td>
<td>$[1.665 \times 10^{-24} \text{ g}, 1.675 \times 10^{-24} \text{ g}]$</td>
</tr>
</tbody>
</table>

Many of the numerical exercises in the text ask that you round the answers to a specified number of decimal places. Our rules for rounding are as follows.
Rules for Rounding a Number (with more than \( n \) Decimal Places) to \( n \) Decimal Places

1. If the digit in the \((n + 1)\)st decimal place is greater than 5, increase the digit in the \(n\)th place by 1. If the digit in the \((n + 1)\)st place is less than 5, leave the \(n\)th digit unchanged.
2. If the digit in the \((n + 1)\)st decimal place is 5 and there is at least one nonzero digit to the right of this 5, increase the digit in the \(n\)th decimal place by 1.
3. If the digit in the \((n + 1)\)st decimal place is 5 and there are no nonzero digits to the right of this 5, then increase the digit in the \(n\)th decimal place by 1 only if this results in an even digit.

The examples in Table 3 illustrate the use of these rules.

<table>
<thead>
<tr>
<th>Number</th>
<th>Rounded to one decimal place</th>
<th>Rounded to three decimal places</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3742</td>
<td>4.4</td>
<td>4.374</td>
</tr>
<tr>
<td>2.0515</td>
<td>2.1</td>
<td>2.052</td>
</tr>
<tr>
<td>2.9925</td>
<td>3.0</td>
<td>2.992</td>
</tr>
</tbody>
</table>

These same rules can be adapted for rounding a result to a specified number of significant digits. As examples of this, we have

- 2347 rounded to two significant digits is \( 2300 = 2.3 \times 10^3 \)
- 2347 rounded to three significant digits is \( 2350 = 2.35 \times 10^3 \)
- 975 rounded to two significant digits is \( 980 = 9.8 \times 10^2 \)
- 0.985 rounded to two significant digits is \( 0.98 = 9.8 \times 10^{-1} \)

In calculator exercises that ask you to round your answers, it’s important that you postpone rounding until the final calculation is carried out. For example, suppose that you are required to determine the hypotenuse \( x \) of the right triangle in Figure 1 to two significant digits. Using the Pythagorean theorem, we have

\[
x = \sqrt{(1.36)^2 + (2.46)^2} = 2.8 \quad \text{using a calculator and rounding the final result to two significant digits}
\]

On the other hand, if we first round each of the given lengths to two significant digits, we obtain

\[
x = \sqrt{(1.4)^2 + (2.5)^2} = 2.9 \quad \text{to two significant digits}
\]

This last result is inappropriate, and we can see why as follows. As Table 4 shows, the maximum possible values for the sides are 1.365 and 2.465, respectively.

<table>
<thead>
<tr>
<th>Number</th>
<th>Range of measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.36</td>
<td>[1.355, 1.365]</td>
</tr>
<tr>
<td>2.46</td>
<td>[2.455, 2.465]</td>
</tr>
</tbody>
</table>
Thus the maximum possible value for the hypotenuse must be
\[ \sqrt{(1.365)^2 + (2.465)^2} = 2.817 \ldots \] calculator display
\[ = 2.8 \] to two significant digits

This shows that the value 2.9 is indeed inappropriate, as we stated previously.

An error that is often made by people working with calculators and approximations is to report a final answer with a greater degree of accuracy than the data warrant. Consider, for example, the right triangle in Figure 2. Using the Pythagorean theorem and a calculator with an eight-digit display, we obtain
\[ h = 3.6055513 \text{ cm} \]

This value for \( h \) is inappropriate, because common sense tells us that the answer should be no more accurate than the data used to obtain that answer. In particular, since the given sides of the triangle apparently were measured only to the nearest tenth of a centimeter, we certainly should not expect any improvement in accuracy for the resulting value of the hypotenuse. An appropriate form for the value of \( h \) here would be \( h = 3.6 \) cm. In general, for calculator exercises in this text that do not specify a required number of decimal places or significant digits in the final results, you should use the following guidelines.

Guidelines for Computing with Approximations

1. For adding and subtracting: Round the final result so that it contains only as many decimal places as there are in the data with the fewest decimal places.
2. For multiplying and dividing: Round the final result so that it contains only as many significant digits as there are in the data with the fewest significant digits.
3. For powers and roots: In computing a power or a root of a real number \( b \), round the result so that it contains as many significant digits as there are in \( b \).

Today’s familiar plus and minus signs were first used in 15th-century Germany as warehouse marks. They indicated when a container held something that weighed over or under a certain standard weight. —Martin Gardner in “Mathematical Games” (Scientific American, June 1977)

In this section we will first list the basic properties for the real number system. Then we will use those properties to prove the familiar rules of algebra for working with signed numbers.

The set of real numbers is closed with respect to the operations of addition and multiplication. This means that when we add or multiply two real numbers, the result (that is, the sum or the product) is again a real number. Some of the other most basic properties and definitions for the real-number system are listed in the following box. In the box, the lowercase letters \( a \), \( b \), and \( c \) denote arbitrary real numbers.

A.2 PROPERTIES OF THE REAL NUMBERS

I do not like \( \times \) as a symbol for multiplication, as it is easily confounded with \( \times \); \ldots often I simply relate two quantities by an interposed dot. \ldots —G. W. Leibniz in a letter dated July 29, 1698
On reading this list of properties for the first time, many students ask the natural question, “Why do we even bother to list such obvious properties?” One reason is that all the other laws of arithmetic and algebra (including the not-so-obvious ones) can be derived from our rather short list. For example, the rule \(0 \cdot a = 0\) can be proved by using the distributive property, as can the rule that the product of two negative numbers is a positive number. We now state and prove those properties and several others.

**THEOREM**

Let \(a\) and \(b\) be real numbers. Then

(a) \(a \cdot 0 = 0\) \quad (b) \(-a = (-1)a\) \quad (c) \((-a) = a\)

(d) \(a(-b) = -ab\) \quad (e) \((-a)(-b) = ab\)

**Proof of Part (a)**

\[
a \cdot 0 = a \cdot (0 + 0) \quad \text{additive identity property}
\]

\[
a \cdot 0 = a \cdot 0 + a \cdot 0 \quad \text{distributive property}
\]

Now since \(a \cdot 0\) is a real number, it has an additive inverse, \(-a \cdot 0\). Adding this to both sides of the last equation, we obtain

\[
a \cdot 0 + [-(a \cdot 0)] = (a \cdot 0 + a \cdot 0) + [-(a \cdot 0)]
\]

\[
a \cdot 0 + [-(a \cdot 0)] = a \cdot 0 + [a \cdot 0 + [-(a \cdot 0)]] \quad \text{associative property of addition}
\]

\[
0 = a \cdot 0 + 0
\]

\[
0 = a \cdot 0 \quad \text{additive identity property}
\]

Thus \(a \cdot 0 = 0\), as we wished to show.
Proof of Part (b)

\[0 = 0 \cdot a\]  
using part (a) and the commutative property of multiplication

\[= [1 + (-1)]a\]  
additive inverse property

\[= 1 \cdot a + (-1)a\]  
distributive property

\[= a + (-1)a\]  
multiplicative identity property

Now, by adding \(-a\) to both sides of this last equation, we obtain

\[-a + 0 = -a + [a + (-1)a]\]

\[-a = (-a + a) + (-1)a\]  
additive identity property and associative property of addition

\[-a = 0 + (-1)a\]  
additive inverse property

\[-a = (-1)a\]  
additive identity property

This last equation asserts that \(-a = (-1)a\), as we wished to show.

Proof of Part (c)

\[-(-a) + (-a) = 0\]  
additive inverse property

By adding \(a\) to both sides of this last equation, we obtain

\[\left[\neg(-a) + (-a)\right] + a = 0 + a\]

\[-(-a) + (-a + a) = a\]  
associative property of addition and additive identity property

\[-(-a) + 0 = a\]  
additive inverse property

\[-(-a) = a\]  
additive identity property

This last equation states that \(-(-a) = a\), as we wished to show.

Proof of Part (d)

\[a(-b) = a(-1)b\]  
using part (b)

\[= [a(-1)]b\]  
associative property of multiplication

\[= (-1)a)b\]  
commutative property of multiplication

\[= (-1)(ab)\]  
associative property of multiplication

\[= -(ab)\]  
using part (b)

Thus \(a(-b) = -ab\), as we wished to show.

Proof of Part (e)

\[(-a)(-b) = -[(-a)b]\]  
using part (d)

\[= -[b(-a)]\]  
commutative property of multiplication

\[= -[-(ba)]\]  
using part (d)

\[= ba\]  
using part (c)

\[= ab\]  
commutative property of multiplication

We’ve now shown that \((-a)(-b) = ab\), as required.
A.3 \( \sqrt{2} \) IS IRRATIONAL

We will use an indirect proof to show that the square root of 2 is an irrational number. The strategy is as follows:

1. We suppose that \( \sqrt{2} \) is a rational number.
2. Using (1) and the usual rules of logic and algebra, we derive a contradiction.
3. On the basis of the contradiction in (2), we conclude that the supposition in (1) is untenable; that is, we conclude that \( \sqrt{2} \) is irrational.

In carrying out the proof, we'll assume that the following three statements are known:

- If \( x \) is an even natural number, then \( x = 2k \) for some natural number \( k \).
- Any rational number can be written in the form \( a/b \), where the integers \( a \) and \( b \) have no common integral factors other than \( \pm 1 \). (In other words, any fraction can be reduced to lowest terms.)
- If \( x \) is a natural number and \( x^2 \) is even, then \( x \) is even.

Our indirect proof now proceeds as follows. Suppose that \( \sqrt{2} \) is a rational number. Then we can write

\[
\sqrt{2} = \frac{a}{b} \quad \text{where } a \text{ and } b \text{ are natural numbers with no common factor other than 1} \tag{1}
\]

Since both sides of equation (1) are positive, we can square both sides to obtain the equivalent equation

\[
2 = \frac{a^2}{b^2}
\]
or

\[
2b^2 = a^2 \tag{2}
\]

Since the left-hand side of equation (2) is an even number, the right-hand side must also be even. But if \( a^2 \) is even, then \( a \) is even, and so

\[
a = 2k \quad \text{for some natural number } k
\]

Using this last equation to substitute for \( a \) in equation (2), we have

\[
2b^2 = (2k)^2 = 4k^2
\]
or

\[
b^2 = 2k^2
\]

Hence (reasoning as before) \( b^2 \) is even, and therefore \( b \) is even. But then we have that both \( b \) and \( a \) are even, contrary to our hypothesis that \( b \) and \( a \) have no common factor other than 1. We conclude from this that equation (1) cannot hold; that is, there is no rational number \( a/b \) such that \( \sqrt{2} = a/b \). Thus \( \sqrt{2} \) is irrational, as we wished to prove.
B.1 REVIEW OF INTEGER EXPONENTS

In basic algebra you learned the exponential notation \( a^n \), where \( a \) is a real number and \( n \) is a natural number. Because that definition is basic to all that follows in this and the next two sections of this appendix, we repeat it here.

**DEFINITION 1  Base and Exponent**

Given a real number \( a \) and a natural number \( n \), we define \( a^n \) by

\[
a^n = a \cdot a \cdot a \cdots a
\]

(n factors)

In the expression \( a^n \) the number \( a \) is the **base**, and \( n \) is the **exponent** or **power** to which the base is raised.

**EXAMPLE 1  Using algebraic notation**

Rewrite each expression using algebraic notation:
(a) \( x \) to the fourth power, plus five;
(b) the fourth power of the quantity \( x \) plus five;
(c) \( x \) plus \( y \), to the fourth power;
(d) \( x \) plus \( y \) to the fourth power.

**SOLUTION**
(a) \( x^4 + 5 \)  \hspace{1cm} (b) \( (x + 5)^4 \)  \hspace{1cm} (c) \( (x + y)^4 \)  \hspace{1cm} (d) \( x + y^4 \)

**CAUTION** Note that parts (c) and (d) of Example 1 differ only in the use of the comma. So for spoken purposes the idea in (c) would be more clearly conveyed by saying “the fourth power of the quantity \( x \) plus \( y \).” In the case of (d), if you read it aloud to another student or your instructor, chances are you’ll be asked, “Do you mean \( x + y^4 \) or do you mean \((x + y)^4\) ?”

**MORAL** Algebra is a precise language; use it carefully. Learn to ask yourself whether what you’ve written will be interpreted in the manner you intended.
In basic algebra the four properties in the following box are developed for working with exponents that are natural numbers. Each of these properties is a direct consequence of the definition of $a^n$. For instance, according to the first property, we have $a^2a^3 = a^5$. To verify that this is indeed correct, we note that

$$a^2a^3 = (aa)(aaa) = a^5$$

### Property Summary

**Properties of Exponents**

<table>
<thead>
<tr>
<th>PROPERTY</th>
<th>EXAMPLES</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $a^m a^n = a^{m+n}$</td>
<td>$a^2a^6 = a^{11}$; $(x + 1)(x + 1)^2 = (x + 1)^3$</td>
</tr>
<tr>
<td>2. $(a^m)^n = a^{mn}$</td>
<td>$(2^3)^4 = 2^{12}$; $[(x + 1)^2]^3 = (x + 1)^6$</td>
</tr>
<tr>
<td>3. $\frac{a^m}{a^n} = \begin{cases} a^{m-n} &amp; \text{if } m &gt; n \ \frac{1}{a^{n-m}} &amp; \text{if } m &lt; n \ 1 &amp; \text{if } m = n \end{cases}$</td>
<td>$\frac{a^5}{a^2} = a^3; \frac{a^2}{a^6} = \frac{1}{a^4}; a^5 = 1$</td>
</tr>
<tr>
<td>4. $(ab)^m = a^m b^m$; $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$</td>
<td>$(2x^3)^3 = 2^3 \cdot (x^3)^3 = 8x^9; \left(\frac{x^3}{y^3}\right)^4 = \frac{x^8}{y^{12}}$</td>
</tr>
</tbody>
</table>

Of course, we assume that all of these expressions make sense; in particular, we assume that no denominator of a fraction is zero.

Now we want to extend our definition of $a^n$ to allow for exponents that are integers but not necessarily natural numbers. We begin by defining $a^0$.

**Definition 2** Zero Exponent

For any nonzero real number $a$, $a^0 = 1$

($0^0$ is not defined.)

**Examples**

- (a) $2^0 = 1$
- (b) $(-\pi)^0 = 1$
- (c) $\left(\frac{3}{1 + a^2 + b^2}\right)^0 = 1$

It’s easy to see the motivation for defining $a^0$ to be 1. Assuming that the exponent zero is to have the same properties as exponents that are natural numbers, we can write

$$a^0 a^n = a^{0+n}$$

That is,

$$a^0 a^n = a^n$$

Now we divide both sides of this last equation by $a^n$ to obtain $a^0 = 1$, which agrees with our definition.

Our next definition (see the box that follows) assigns a meaning to the expression $a^{-n}$ when $n$ is a natural number. Again, it’s easy to see the motivation for this definition. We would like to have

$$a^n a^{-n} = a^{n+(-n)} = a^0 = 1$$

That is,

$$a^n a^{-n} = 1$$
Now we divide both sides of this last equation by $a^n$ to obtain $a^{-n} = 1/a^n$, in agreement with Definition 3.

**DEFINITION 3** Negative Exponent

For any nonzero real number $a$ and natural number $n$,
\[ a^{-n} = \frac{1}{a^n} \]

**EXAMPLES**

(a) $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

(b) $\left(\frac{1}{10}\right)^{-1} = \frac{1}{\left(\frac{1}{10}\right)^1} = 10$

(c) $x^{-2} = \frac{1}{x^2}$

(d) $(a^2b)^{-3} = \frac{1}{(a^2b)^3} = \frac{1}{a^6b^3}$

(e) $\frac{1}{2^{-3}} = \frac{1}{1/2^3} = 2^3 = 8$

It can be shown that the four properties of exponents that we listed earlier continue to hold now for all integer exponents. We make use of this fact in the next three examples.

**EXAMPLE 2** Using properties of exponents

Simplify the following expression. Write the answer in such a way that only positive exponents appear.

\[ (a^2b^3)^2(a^{-5}b)^{-1} \]

**SOLUTION**

**FIRST METHOD**

\[ (a^2b^3)^2(a^{-5}b)^{-1} = (a^2b^3)^2 \cdot \frac{1}{a^{-5}b} \]

\[ = a^{4}b^{6} \cdot \frac{b^{-1}}{a^{-5}} = \frac{b^5}{a} \]

**ALTERNATIVE METHOD**

\[ (a^2b^3)^2(a^{-5}b)^{-1} = (a^2b^3)(a^{-5}b^{-1}) \]

\[ = a^{4-5}b^{3-1} = a^{-1}b^2 \]

\[ = \frac{b^5}{a} \text{ as obtained previously} \]

**EXAMPLE 3** Using properties of exponents

Simplify the following expression, writing the answer so that negative exponents are not used.

\[ \left(\frac{a^{-5}b^2c^0}{a^1b^{-1}}\right)^3 \]
SOLUTION
We show two solutions. The first makes immediate use of the property \((a^m)^n = a^{mn}\). In the second solution we begin by working within the parentheses.

FIRST SOLUTION
\[
\left( \frac{a^3 b^7 c^0}{a^2 b^4} \right)^3 = \frac{a^{-13} b^6}{a^9 b^{-3}}
\]
\[
= \frac{b^{6-(-3)}}{a^{9-(-15)}} = \frac{b^9}{a^{24}}
\]

ALTERNATIVE SOLUTION
\[
\left( \frac{a^3 b^7 c^0}{a^2 b^4} \right)^3 = \left( \frac{b^3}{a^8} \right)^3
\]
\[
= \frac{b^9}{a^{24}}
\]

EXAMPLE 4 Simplifying with variable exponents

Simplify the following expressions, writing the answers so that negative exponents are not used. (Assume that \(p\) and \(q\) are natural numbers.)

(a) \(\frac{a^{4p+q}}{a^{p-q}}\)  
(b) \((a^p b^q)^2 (a^{3p} b^{-q})^{-1}\)

SOLUTION
(a) \(\frac{a^{4p+q}}{a^{p-q}} = a^{4p+q-(p-q)} = a^{3p+2q}\)
(b) \((a^p b^q)^2 (a^{3p} b^{-q})^{-1} = (a^{2p} b^{2q})(a^{-3p} b^q)\)
\[
= a^{-p} b^{3q}
\]
\[
= \frac{b^{3q}}{a^p}
\]

EXAMPLE 5 Simplifying numerical quantities involving exponents

Use the properties of exponents to compute the quantity \(\frac{2^{10} \cdot 3^{13}}{27 \cdot 6^{12}}\).

SOLUTION
\[
\frac{2^{10} \cdot 3^{13}}{27 \cdot 6^{12}} = \frac{2^{10} \cdot 3^{13}}{(3^3)(3^2)^{12}}
\]
\[
= \frac{2^{10} \cdot 3^{13}}{3^3 \cdot 3^{12} \cdot 2^{12}} = \frac{2^{10} \cdot 3^{13}}{3^{15} \cdot 2^{12}}
\]
\[
= \frac{1}{(3^{15-13})(2^{12-10})} = \frac{1}{3^2 \cdot 2^2} = \frac{1}{36}
\]

As an application of some of the ideas in this section, we briefly discuss scientific notation, which is a convenient form for writing very large or very small
numbers. Such numbers occur often in the sciences. For instance, the speed of light in a vacuum is

\[ 29,979,000,000 \text{ cm/sec} \]

As written, this number would be awkward to work with in calculating. In fact, the number as written cannot even be displayed on some hand-held calculators, because there are too many digits. To write the number 29,979,000,000 (or any positive number) in scientific notation, we express it as a number between 1 and 10, multiplied by an appropriate power of 10. That is, we write it in the form

\[ a \times 10^n \quad \text{where } 1 \leq a < 10 \text{ and } n \text{ is an integer} \]

According to this convention, the number 4.03 \times 10^6 is in scientific notation, but the same quantity written as 40.3 \times 10^5 is not in scientific notation. In order to convert a given number into scientific notation, we’ll rely on the following two-step procedure.

**To Express a Number Using Scientific Notation**

1. First move the decimal point until it is to the immediate right of the first nonzero digit.
2. Then multiply by 10^a or 10^{-n}, depending on whether the decimal point was moved n places to the left or to the right, respectively.

For example, to express the number 29,979,000,000 in scientific notation, first we move the decimal point 10 places to the left so that it’s located between the 2 and the 9, then we multiply by 10^{10}. The result is

\[ 29,979,000,000 = 2.9979000000 \times 10^{10} \]

or, more simply,

\[ 29,979,000,000 = 2.9979 \times 10^{10} \]

As additional examples we list the following numbers expressed in both ordinary and scientific notation. For practice, you should verify each conversion for yourself using our two-step procedure.

- \[ 55,708 = 5.5708 \times 10^4 \]
- \[ 0.000099 = 9.9 \times 10^{-5} \]
- \[ 0.0000002 = 2 \times 10^{-7} \]

All scientific calculators have keys for entering numbers in scientific notation. (Indeed, many calculators will convert numbers into scientific notation for you.) If you have questions about how your own calculator operates with respect to scientific notation, you should consult the user’s manual. We’ll indicate only one example here, using a Texas Instruments graphing calculator. The keystrokes on other brands are quite similar. Consider the following number expressed in scientific notation: 3.159 \times 10^{20}. (This gargantuan number is the diameter, in miles, of a galaxy at the center of a distant star cluster known as Abell 2029.) The sequence of keystrokes for entering this number on the Texas Instruments calculator is

\[ 3.519 \ EE \ 20 \]
EXERCISE SET B.1

A

In Exercises 1–4, evaluate each expression using the given value of x.
1. \(2x^3 - x + 4; x = -2\)
2. \(1 - 2x^2 + 3x^3; x = -1\)
3. \(\frac{1 - 2x^2}{1 + 2x^3}; x = \frac{1}{2}\)

In Exercises 5–16, use the properties of exponents to simplify each expression.
5. (a) \(a^{12}\)
   (b) \((a + 1)^3(a + 1)^{12}\)
   (c) \((a + 1)^4(a + 1)^3\)
6. (a) \(3y^3 - 2y^2\)
   (b) \((x^2y)^3 - (x^3y)^2\)
   (c) \((x^2)^4 - (x^3)^2\)
7. (a) \(\frac{x^{12}}{x^8}\)
   (b) \(\frac{x^{10}}{x^2}\)
   (c) \(\frac{x^5}{x^2}\)
8. (a) \((x^2)^3\)
   (b) \(y^2 + y + 1\)
   (c) \((y + 1)^2\)
9. (a) \(\frac{x^2 + 3}{x^2 + 3}\)
   (b) \(\frac{x^2 + 3}{x^2 + 3}\)
   (c) \(\frac{x^2 + 3}{x^2 + 3}\)
10. (a) \(\frac{y^{15}}{y^2}\)
    (b) \(\frac{y^9}{y^2}\)
    (c) \(\frac{y^{15}}{(y^2 + 3)^5}\)
11. (a) \(\frac{x^6}{x}\)
    (b) \(\frac{x}{x^6}\)
    (c) \(\frac{x^2 + 3x - 2}{x^2 + 3x - 2}\)
12. (a) \(2x^2 - 1\)
    (b) \(2x^2 - 1\)
    (c) \(2x^2 - 1\)
13. (a) \(4x^2y\)
    (b) \(4x^2y\)
    (c) \(4x^2y\)
14. (a) \(2(x - 1)^2 - (x - 1)^2\)
    (b) \(2(x - 1)^2 - (x - 1)^2\)
    (c) \(2(x - 1)^2 - (x - 1)^2\)
15. (a) \(3x^2 + 3\)
    (b) \(3x^2 + 3\)
    (c) \(3x^2 + 3\)
16. (a) \(64^9\)
    (b) \(64^9\)
    (c) \(64^9\)

B.1 Review of Integer Exponents

In Exercises 39–42, use the properties of exponents in computing each quantity, as in Example 5. (The point here is to do as little arithmetic as possible.)
39. \(\frac{2^8 \cdot 3^{15}}{9 \cdot 3^{10} \cdot 12}\)
40. \(\frac{2^{12} \cdot 5^{13}}{10^{12}}\)
41. \(\frac{24^5}{32 \cdot 12^8}\)
42. \(\left(\frac{144 \cdot 125}{2^3 \cdot 3^2}\right)^{-1}\)

For Exercises 43–50, express each number in scientific notation.
43. The average distance (in miles) from the Earth to the Sun: \(92,900,000\)
44. The average distance (in miles) from the planet Pluto to the Sun: \(3,666,000,000\)
45. The average orbital speed (in miles per hour) of the Earth: \(66,800\)
46. The average orbital speed (in miles per hour) of the planet Mercury: \(107,300\)
47. The average distance (in miles) from the Sun to the nearest star: \(25,000,000,000,000,000,000\)
48. The equatorial diameter (in miles) of
   (a) Mercury: \(3031\)
   (b) the Earth: \(7927\)
   (c) Jupiter: \(88,733\)
   (d) the Sun: \(865,000\)
49. The time (in seconds) for light to travel
(a) one foot: 0.000000001
(b) across an atom: 0.000000000000000001
(c) across the nucleus of an atom: 0.000000001

50. The mass (in grams) of
(a) a proton: 0.00000000000000000000000167
(b) an electron: 0.00000000000000000000000911

### B.2 REVIEW OF nTH ROOTS

In this section we generalize the notion of square root that you studied in elementary algebra. The new idea is that of an nth root; the definition and some basic examples are given in the box that follows.

#### DEFINITION 1 nth Roots

Let \( n \) be a natural number. If \( a \) and \( b \) are real numbers and
\[
    a^n = b
\]
then we say that \( a \) is an \textbf{nth root} of \( b \).

When \( n = 2 \) and when \( n = 3 \), we refer to the roots as \textbf{square roots} and \textbf{cube roots}, respectively.

#### EXAMPLES

Both 3 and \( \sqrt[2]{9} \) are square roots of 9 because \( 3^2 = 9 \) and \((-3)^2 = 9\).

Both 2 and \(-2\) are fourth roots of 16 because \( 2^4 = 16 \) and \((-2)^4 = 16\).

2 is a cube root of 8 because \( 2^3 = 8 \).

\(-3\) is a fifth root of \(-243\) because \((-3)^5 = -243\).

The symbol \( \sqrt{} \) is called a \textbf{radical sign}, and the number within the radical sign is the \textbf{radicand}. The natural number \( n \) used in the notation \( \sqrt[n]{b} \) is called the \textbf{index} of the radical. For square roots, as you know from basic algebra, we suppress the index and simply write \( \sqrt{} \) rather than \( \sqrt[n]{b} \). So, for example, \( \sqrt[2]{25} = 5 \).

On the other hand, as we saw in the examples, cube roots, fifth roots, and in fact, all \textit{odd} roots occur singly, not in pairs. In these cases, we again use the notation \( \sqrt[5]{b} \) for the \( n \)th root. The definition and examples in the following box summarize our discussion up to this point.

Hindu mathematicians first recognized negative roots, and the two square roots of a positive number, \( \ldots \) though they were suspicious also. —David Wells in \textit{The Penguin Dictionary of Curious and Interesting Numbers} (Harmondsworth, Middlesex, England: Penguin Books, Ltd., 1986)
DEFINITION 2  Principal $n$th Root

1. Let $n$ be a natural number. If $a$ and $b$ are nonnegative real numbers, then

$$\sqrt[n]{b} = a \quad \text{if and only if} \quad b = a^n$$

The number $a$ is the **principal** $n$th root of $b$.

2. If $a$ and $b$ are negative and $n$ is an odd natural number, then

$$\sqrt[n]{b} = a \quad \text{if and only if} \quad b = a^n$$

There are five properties of $n$th roots that are frequently used in simplifying certain expressions. The first four are similar to the properties of square roots that are developed in elementary algebra. For reference we list these properties side by side in the following box. Property 5 is listed here only for the sake of completeness; we’ll postpone discussing it until Appendix B.3.

<table>
<thead>
<tr>
<th>PROPERTY SUMMARY</th>
<th>Properties of $n$th Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1.</strong> $(\sqrt[n]{x})^n = x$</td>
<td><strong>CORRESPONDING PROPERTIES FOR SQUARE ROOTS</strong></td>
</tr>
<tr>
<td><strong>2.</strong> $\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$</td>
<td>$(\sqrt{x})^2 = x$</td>
</tr>
<tr>
<td><strong>3.</strong> $\sqrt[n]{x/y} = \sqrt[n]{x} / \sqrt[n]{y}$</td>
<td>$\sqrt{x/y} = \sqrt{x} / \sqrt{y}$</td>
</tr>
<tr>
<td><strong>4.</strong> If $n$ even: $\sqrt[n]{x^2} =</td>
<td>x</td>
</tr>
<tr>
<td><strong>5.</strong> $\sqrt[n]{x^n} = x$</td>
<td>$\sqrt{n} \sqrt{x} = \sqrt[2]{x}$</td>
</tr>
<tr>
<td><strong>5.</strong> $\sqrt[n]{m/n} = m/n \sqrt[n]{x}$</td>
<td>$\sqrt{n} \sqrt{x} = \sqrt[2]{x}$</td>
</tr>
</tbody>
</table>

Our immediate use for these properties will be in simplifying expressions involving $n$th roots. In general, we try to factor the expression under the radical so that one factor is the largest perfect $n$th power that we can find. Then we apply Property 2 or 3. For instance, the expression $\sqrt{72}$ is simplified as follows:

$$\sqrt{72} = \sqrt{(36)(2)} = \sqrt{36} \sqrt{2} = 6 \sqrt{2}$$

In this procedure we began by factoring 72 as $(36)(2)$. Note that 36 is the largest factor of 72 that is a perfect square. If we were to begin instead with a different factorization, for example, $72 = (9)(8)$, we could still arrive at the same answer, but it would take longer. (Check this for yourself.) As another example, let’s simplify $\sqrt[3]{40}$. First, what (if any) is the largest perfect-cube factor of 40? The first few perfect cubes are

$$1^3 = 1 \quad 2^3 = 8 \quad 3^3 = 27 \quad 4^3 = 64$$
so we see that 8 is a perfect-cube factor of 40, and we write

\[
\sqrt{40} = \sqrt{(8)(5)} = \sqrt{8} \sqrt{5} = 2\sqrt{5}
\]

**EXAMPLE 1** Simplifying square roots

Simplify:  
\(a\) \(\sqrt{12} + \sqrt{75}\);  \(b\) \(\sqrt{\frac{162}{49}}\)

**SOLUTION**

\(a\) \(\sqrt{12} + \sqrt{75} = \sqrt{(4)(3)} + \sqrt{(25)(3)} = 2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}\)

\(b\) \(\sqrt{\frac{162}{49}} = \frac{\sqrt{162}}{\sqrt{49}} = \frac{9\sqrt{2}}{7}\)

**EXAMPLE 2** Simplifying \(n\)th roots

Simplify: \(\sqrt{16} + \sqrt{250} - \sqrt{128}\).

**SOLUTION**

\(\sqrt{16} + \sqrt{250} - \sqrt{128} = 4\sqrt{2} + 5\sqrt{2} - 4\sqrt{2} = 3\sqrt{2}\)

**EXAMPLE 3** Simplifying radicals containing variables

Simplify each of the following expressions by removing the largest possible perfect-square or perfect-cube factor from within the radical:

\(a\) \(\sqrt{8x^2}\);

\(b\) \(\sqrt{8x^2}\), where \(x \geq 0\);

\(c\) \(\sqrt{18a^2}\), where \(a \geq 0\);

\(d\) \(\sqrt{16y^3}\).

**SOLUTION**

\(a\) \(\sqrt{8x^2} = \sqrt{(4)(2)(x^2)} = 2\sqrt{2}|x|\)

\(b\) \(\sqrt{8x^2} = \sqrt{4\sqrt{2}x^2} = 2\sqrt{2}x\) because \(x \geq 0\)

\(c\) \(\sqrt{18a^2} = \sqrt{(9a^2)(2a)} = 3a\sqrt{2a}\)

\(d\) \(\sqrt{16y^3} = \sqrt{8y^2\cdot2y} = 2y\sqrt{2y^2}\)
**EXAMPLE 4 Simplifying nth roots containing variables**

Simplify each of the following, assuming that $a$, $b$, and $c$ are positive:

(a) $\sqrt[4]{8abc^2}$; (b) $\sqrt[4]{\frac{32a^5b^2}{c^8}}$.

**SOLUTION**

(a) $\sqrt[4]{8abc^2} = \sqrt[4]{(4b^2c^4)(2ac)} = \sqrt[4]{4b^2c^4\sqrt[4]{2ac}} = 2bc\sqrt[4]{2ac}$

(b) $\sqrt[4]{\frac{32a^5b^2}{c^8}} = \frac{\sqrt[4]{32a^5b^2}}{\sqrt[4]{c^8}} = \frac{\sqrt[4]{16a^4b^8\sqrt[4]{2a^2bc}}}{c^2} = \frac{2ab\sqrt[4]{2a^2bc}}{c^2}$

In Examples 1 through 4 we used the definitions and properties of $n$th roots to simplify certain expressions. The box that follows shows some common errors to avoid in working with roots. Use the error box to test yourself: Cover up the columns labeled “Correction” and “Comment,” and try to decide for yourself where the errors lie.

<table>
<thead>
<tr>
<th>Error</th>
<th>Correction</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt[4]{16} = \pm 2$</td>
<td>$\sqrt[4]{16} = 2$</td>
<td>Although $-2$ is one of the fourth roots of 16, it is not the principal fourth root. The notation $\sqrt[4]{16}$ is reserved for the principal fourth root.</td>
</tr>
<tr>
<td>$\sqrt[5]{25} = -5$</td>
<td>$\sqrt[5]{25} = 5$</td>
<td>Although $-5$ is one of the square roots of 25, it is not the principal square root.</td>
</tr>
<tr>
<td>$\sqrt[4]{a+b} = \sqrt[4]{a} + \sqrt[4]{b}$</td>
<td>The expression $\sqrt[4]{a+b}$ cannot be simplified.</td>
<td>The properties of roots differ with respect to addition versus multiplication. For $\sqrt[4]{ab}$ we do have the simplification $\sqrt[4]{ab} = \sqrt[4]{a} \sqrt[4]{b}$ (assuming that $a$ and $b$ are nonnegative).</td>
</tr>
<tr>
<td>$\sqrt[4]{a+b} = \sqrt[4]{a} + \sqrt[4]{b}$</td>
<td>The expression $\sqrt[4]{a+b}$ cannot be simplified.</td>
<td>For $\sqrt[4]{ab}$ we do have $\sqrt[4]{ab} = \sqrt[4]{a} \sqrt[4]{b}$.</td>
</tr>
</tbody>
</table>

There are times when it is convenient to rewrite fractions involving radicals in alternative forms. Suppose, for example, that we want to rewrite the fraction $\frac{5}{\sqrt{3}}$ in an equivalent form that does not involve a radical in the denominator. This is called **rationalizing the denominator**. The procedure here is to multiply by 1 in this way:

$$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot 1 = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$$

That is, $\frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$, as required.
To rationalize a denominator of the form $a + \sqrt{b}$, we need to multiply the fraction not by $\sqrt{b}/\sqrt{b}$, but rather by $(a - \sqrt{b})/(a - \sqrt{b}) (= 1)$. To see why this is necessary, notice that

$$(a + \sqrt{b})\sqrt{b} = a\sqrt{b} + b$$

which still contains a radical, whereas

$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - a\sqrt{b} + a\sqrt{b} - b = a^2 - b$$

which is free of radicals. Note that we multiply $a + \sqrt{b}$ by $a - \sqrt{b}$ to obtain an expression that is free of square roots by taking advantage of the form of a difference of two squares. Similarly, to rationalize a denominator of the form $a - \sqrt{b}$, we multiply the fraction by $(a + \sqrt{b})/(a + \sqrt{b})$. (The quantities $a + \sqrt{b}$ and $a - \sqrt{b}$ are said to be conjugates of each other.) The next two examples make use of these ideas.

**EXAMPLE 5** Rationalizing a square root denominator

Simplify: $\frac{1}{\sqrt{2}} - 3\sqrt{50}$.

**SOLUTION**

First, we rationalize the denominator in the fraction $1/\sqrt{2}$:

$$\frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Next, we simplify the expression $3\sqrt{50}$:

$$3\sqrt{50} = 3\sqrt{25\cdot2} = 3\sqrt{25}\sqrt{2} = (3)(5)\sqrt{2} = 15\sqrt{2}$$

Now, putting things together, we have

$$\frac{1}{\sqrt{2}} - 3\sqrt{50} = \frac{\sqrt{2}}{2} - 15\sqrt{2} = \frac{\sqrt{2}}{2} - \frac{30\sqrt{2}}{2} = \frac{\sqrt{2} - 30\sqrt{2}}{2} = \frac{(1 - 30)\sqrt{2}}{2} = \frac{-29\sqrt{2}}{2}$$

**EXAMPLE 6** Using the conjugate to rationalize a denominator

Rationalize the denominator in the expression $\frac{4}{2 + \sqrt{3}}$.

**SOLUTION**

We multiply by 1, writing it as $\frac{2 - \sqrt{3}}{2 - \sqrt{3}}$. 

$$\frac{4}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = \frac{4(2 - \sqrt{3})}{(2 + \sqrt{3})(2 - \sqrt{3})} = \frac{4(2 - \sqrt{3})}{4 - 3} = 4(2 - \sqrt{3})$$
B.2 Review of nth Roots

Note: Check for yourself that multiplying the original fraction \( \frac{2 + \sqrt{3}}{2 + \sqrt{3}} \) does not eliminate radicals in the denominator.

In the next example we are asked to rationalize the numerator rather than the denominator. This is useful at times in calculus.

EXAMPLE 7 Rationalizing a numerator using the conjugate

Rationalize the numerator of: \( \frac{\sqrt{x} - \sqrt{3}}{x - 3} \), where \( x \geq 0, x \neq 3 \).

SOLUTION
\[
\frac{\sqrt{x} - \sqrt{3}}{x - 3} \cdot \frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}} = \frac{\sqrt{x} - \sqrt{3}}{x - 3} \cdot \frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}}
\]
\[
= \frac{(\sqrt{x})^2 - (\sqrt{3})^2}{(x - 3)(\sqrt{x} + \sqrt{3})}
\]
\[
= \frac{x - 3}{\sqrt{x} + \sqrt{3}}
\]

The strategy for rationalizing numerators or denominators involving nth roots is similar to that used for square roots. To rationalize a numerator or a denominator involving an nth root, we multiply the numerator or denominator by a factor that yields a product that itself is a perfect nth power. The next example displays two instances of this.

EXAMPLE 8 Rationalizing denominators containing nth roots

(a) Rationalize the denominator of: \( \frac{6}{\sqrt{7}} \).

(b) Rationalize the denominator of: \( \frac{ab}{\sqrt[3]{a^2 b^2}} \), where \( a > 0, b > 0 \).

SOLUTION
(a) \[
\frac{6}{\sqrt{7}} \cdot 1 = \frac{6}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}
\]
\[
= \frac{6\sqrt{7}}{7}
\]
\[
= \frac{6\sqrt{49}}{7} \quad \text{as required}
\]
Finally, just as we used the difference of squares formula to motivate the method of rationalizing certain two-term denominators containing a square root term, we can use the difference or sum of cubes formula (see Appendix B.4) to rationalize a denominator involving a cube root term. In particular, in the sum of cubes formula $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$, the expression $(x^2 - xy + y^2)$ is the conjugate of the expression $x + y$.

**Example 9** Using the conjugate to rationalize a denominator involving a cube root

Rationalize the denominator in the expression $\frac{1}{\sqrt[3]{a} + \sqrt[3]{b}}$.

**Solution** If we let $x = \sqrt[3]{a}$ and $y = \sqrt[3]{b}$, the conjugate of the denominator $x + y = \sqrt[3]{a} + \sqrt[3]{b}$ is

$$x^3 - y^3 = (\sqrt[3]{a})^3 - (\sqrt[3]{b})^3 = \sqrt[3]{a^3} - \sqrt[3]{b^3}$$

where, for example $(\sqrt[3]{a})^3 = \sqrt[3]{a^3} = \sqrt[3]{a}^2$. Then

$$\frac{1}{\sqrt[3]{a} + \sqrt[3]{b}} = \frac{1}{\sqrt[3]{a^3} + \sqrt[3]{b^3}} = \frac{\sqrt[3]{a}^2 - \sqrt[3]{ab} + \sqrt[3]{b}^2}{(\sqrt[3]{a})^3 + (\sqrt[3]{b})^3} = \frac{\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2}}{a + b}$$

**Exercise Set B.2**

**A**

In Exercises 1–8, determine whether each statement is TRUE or FALSE.

1. $\sqrt{81} = -9$
2. $\sqrt{256} = 16$
3. $-\sqrt{100} = -10$
4. $\sqrt{49} = -7$
5. $\sqrt{4/3} = 2/\sqrt{3}$
6. $\sqrt{10} + 6 = \sqrt{10} + 6$
7. $(\sqrt{10})^3 = 10$

For Exercises 9–18, evaluate each expression. If the expression is undefined (i.e., does not represent a real number), say so.

9. (a) $\sqrt{-64}$
   (b) $\sqrt{-64}$
10. (a) $\sqrt{32}$
    (b) $\sqrt{-32}$
11. (a) $\sqrt{8/125}$
    (b) $\sqrt{-8/125}$
12. (a) $\sqrt{-1/1000}$
    (b) $\sqrt{-1/1000}$
13. (a) $\sqrt{-16}$
    (b) $\sqrt{-16}$
14. (a) $\sqrt{16}$
    (b) $\sqrt{-16}$
15. (a) $\sqrt{256/81}$
    (b) $\sqrt{-27/125}$
16. (a) $\sqrt{64}$
    (b) $\sqrt{-64}$
17. (a) $\sqrt{-32}$
    (b) $\sqrt{-32}$
18. (a) $\sqrt{(-10)^2}$
    (b) $\sqrt{(-10)^3}$

In Exercises 19–44, simplify each expression. Unless otherwise specified, assume that all letters in Exercises 35–44 represent positive numbers.

19. (a) $\sqrt{18}$
    (b) $\sqrt{33}$
20. (a) $\sqrt{150}$
    (b) $\sqrt{375}$
21. (a) $\sqrt{98}$
    (b) $\sqrt{-64}$
22. (a) $\sqrt{27}$
    (b) $\sqrt{-108}$
B.3 Review of Rational Exponents

\[
\begin{align*}
23. & \quad \sqrt{25/4} \\
& \quad \sqrt{16}/25 \\
25. & \quad \sqrt{2} + \sqrt{3} \\
& \quad \sqrt{2} + \sqrt{16} \\
27. & \quad 4\sqrt{30} - 3\sqrt{128} \\
& \quad \sqrt{32} + \sqrt{162} \\
28. & \quad \sqrt{3} - \sqrt{12} + \sqrt{54} \\
& \quad \sqrt{-2} + \sqrt{-64} - \sqrt{486} \\
29. & \quad \sqrt{0.09} \\
& \quad \sqrt{0.08} \\
30. & \quad \sqrt{-2} + \sqrt{9} \\
& \quad \sqrt{81/121} - \sqrt{-8/1331} \\
31. & \quad 4\sqrt{3} - 8\sqrt{5} + 2\sqrt{6} \\
33. & \quad \sqrt{\sqrt{64}} \\
35. & \quad \sqrt{36a^2}, \quad \text{where } y < 0 \\
& \quad \sqrt{36y^2}, \quad \text{where } y < 0 \\
37. & \quad \sqrt{a^2b^2}, \quad \sqrt{a^2b^2} \\
& \quad \sqrt{125a^2}, \quad \sqrt{125a^2} \\
39. & \quad \sqrt{72a^2b^2}, \quad \sqrt{72a^2b^2} \\
41. & \quad \sqrt{16a^2b^2} \\
43. & \quad \sqrt{16a^2b^2}/c^2 \\
46. & \quad 3/\sqrt{3} \\
48. & \quad 3/\sqrt{3} \\
50. & \quad \sqrt{2}/(1 - \sqrt{2}) \\
52. & \quad \sqrt{a + \sqrt{b}}/(\sqrt{a} - \sqrt{b}) \\
54. & \quad \sqrt{3}/\sqrt{5} - \sqrt{4} \\
56. & \quad 4/\sqrt{16} \\
58. & \quad \sqrt{5}/\sqrt{5} \\
60. & \quad 3/\sqrt{27a^2b^2} \\
61. & \quad 2/(\sqrt{3} + 1) \\
& \quad 1/(2\sqrt{5} + 1) \\
62. & \quad \sqrt{2}/(\sqrt{3} - 1) \\
& \quad \sqrt{1}/(2\sqrt{5} - 1) \\
63. & \quad \sqrt{a}/(\sqrt{a} - 2) \\
& \quad \sqrt{a}/(\sqrt{a} - 2) \\
64. & \quad \sqrt{a}/(\sqrt{a} + \sqrt{a}) \\
& \quad \sqrt{a}/(\sqrt{a} + \sqrt{a}) \\
65. & \quad (\sqrt{a} + \sqrt{b})/(\sqrt{a} - \sqrt{b}) \\
& \quad (\sqrt{a} + \sqrt{a})/(\sqrt{a} - \sqrt{a}) \\
66. & \quad (\sqrt{a} + 2\sqrt{y})/(\sqrt{a} - 2\sqrt{y}) \\
& \quad (\sqrt{a} - 2\sqrt{y})/(\sqrt{a} + 2\sqrt{y}) \\
67. & \quad -2/(\sqrt{a} + \sqrt{b} - \sqrt{a}) \\
& \quad 1/(\sqrt{a} - \sqrt{b}) \\
68. & \quad 1/(\sqrt{a} + \sqrt{b} - \sqrt{a}) \\
69. & \quad 1/(\sqrt{a} - \sqrt{b}) \\
70. & \quad 1/(\sqrt{a} - \sqrt{b}) \\
71. & \quad (\sqrt{a} + 3)/(4y - 9) \\
72. & \quad (\sqrt{a} + 3)/(4y - 9) \\
73. & \quad (\sqrt{a} + 3)/(4y - 9) \\
74. & \quad (\sqrt{a} + 3)/(4y - 9) \\
75. & \quad (\sqrt{a} + 3)/(4y - 9) \\
76. & \quad (\sqrt{a} + 3)/(4y - 9) \\
77. & \quad \sqrt{a + b} = \sqrt{a} + \sqrt{b} \\
78. & \quad \sqrt{x^2 + y^2} = x + y \\
79. & \quad \sqrt{a + b} = \sqrt{a} + \sqrt{b} \\
80. & \quad \sqrt{p^2 + q^2} = p + q \\
\end{align*}
\]

In Exercises 71–76, rationalize the denominator.

In Exercises 77–80, refer to the following table. The left-hand column of the table lists four errors to avoid in working with expressions containing radicals. In each case, give a numerical example showing that the expressions on each side of the equation are, in general, not equal. Use a calculator as necessary.

\begin{center}
\begin{tabular}{|l|}
\hline
\textbf{Errors to avoid} \\
\hline
\textbf{Numerical example} \\
showing that the \\
formula is not, \\
in general, valid \\
\hline
\end{tabular}
\end{center}

\begin{align*}
77. & \quad \sqrt{a + b} = \sqrt{a} + \sqrt{b} \\
78. & \quad \sqrt{x^2 + y^2} = x + y \\
79. & \quad \sqrt{a + b} = \sqrt{a} + \sqrt{b} \\
80. & \quad \sqrt{p^2 + q^2} = p + q \\
\end{align*}

B

81. (a) Use a calculator to evaluate \(\sqrt{8 - 2\sqrt{7}}\) and \(\sqrt{7} - 1\). What do you observe?

(b) Prove that \(\sqrt{8 - 2\sqrt{7}} = \sqrt{7} - 1\). \textit{Hint:} In view of the definition of a principal square root, you need to check that \(\sqrt{7} - 1\)^2 = 8 - 2\sqrt{7}.

82. Use a calculator to provide empirical evidence indicating that both of the following equations may be correct:

\[
\frac{2 - \sqrt{3}}{\sqrt{2} - \sqrt{2} - \sqrt{3}} + \frac{2 + \sqrt{3}}{\sqrt{2} + \sqrt{2} + \sqrt{3}} = \sqrt{7}
\]

The point of this exercise is to remind you that as useful as calculators may be, there is still the need for proofs in mathematics. In fact, it can be shown that the first equation is indeed correct, but the second is not.
We can use the concept of an nth root to give a meaning to fractional exponents that is useful and, at the same time, consistent with our earlier work. First, by way of motivation, suppose that we want to assign a value to $5^{1/3}$. Assuming that the usual properties of exponents continue to apply here, we can write

$$\sqrt[3]{5} = 5^{1/3}$$

That is,

$$5^{1/3} = 5$$

or,

$$5^{1/3} = \sqrt[3]{5}$$

By replacing 5 and 3 with $b$ and $n$, respectively, we can see that we want to define $b^{1/n}$ to mean $\sqrt[n]{b}$. Also, by thinking of $b^{m/n}$ as $(b^{1/n})^m$, we see that the definition for $b^{m/n}$ ought to be $(\sqrt[n]{b})^m$. These definitions are formalized in the box that follows.

**DEFINITION**

Rational Exponents

1. Let $b$ denote a real number and $n$ a natural number. We define $b^{1/n}$ by

$$b^{1/n} = \sqrt[n]{b}$$

(If $n$ is even, we require that $b \geq 0$.)

2. Let $m/n$ be a rational number reduced to lowest terms. Assume that $n$ is positive and that $\sqrt[n]{b}$ exists. Then

$$b^{m/n} = (\sqrt[n]{b})^m$$

or, equivalently,

$$b^{m/n} = \sqrt[n]{b^m}$$

It can be shown that the four properties of exponents that we listed in Appendix B.1 (on page A-9) continue to hold for rational exponents in general. In fact, we’ll take this for granted rather than follow the lengthy argument needed for its verification. We will also assume that these properties apply to irrational exponents. For instance, we have

$$(2^{\sqrt{3}})^{\sqrt{2}} = 2^2 = 4$$

(The definition of irrational exponents is discussed in Section 5.1.) In the next three examples we display the basic techniques for working with rational exponents.

**EXAMPLE 1** Evaluating rational exponents

Simplify each of the following quantities. Express the answers using positive exponents. If an expression does not represent a real number, say so.

(a) $49^{1/2}$  
(b) $-49^{1/2}$  
(c) $(-49)^{1/2}$  
(d) $49^{-1/2}$

**SOLUTION**

(a) $49^{1/2} = \sqrt{49} = 7$

(b) $-49^{1/2} = -(49^{1/2}) = -\sqrt{49} = -7$
(c) The quantity $(-49)^{1/2}$ does not represent a real number because there is no real number $x$ such that $x^2 = -49$.

(d) $49^{-1/2} = \sqrt{49^{-1}} = \sqrt{1/49} = \frac{\sqrt{1}}{\sqrt{49}} = \frac{1}{7}$

Alternatively, we have

$$49^{-1/2} = (49^{1/2})^{-1} = 7^{-1} = \frac{1}{7}$$

Example 2: Simplifying expressions containing rational exponents

Simplify each of the following. Write the answers using positive exponents. (Assume that $a > 0$.)

(a) $(5a^{2/3})(4a^{3/4})$

(b) $\sqrt[10]{16a^{1/3}}$

(c) $(x^2 + 1)^{1/5}(x^2 + 1)^{4/5}$

Example 3: Evaluating rational exponents

Simplify: (a) $32^{-2/5}$, (b) $(-8)^{4/3}$.

Example

Rational exponents can be used to simplify certain expressions containing radicals. For example, one of the properties of $n$th roots that we listed but did not discuss in Appendix B.2 is $\sqrt[n]{x} = x^{\frac{m}{n}}$. Using exponents, it is easy to verify this property. We have
\[ \sqrt[\theta]{\sqrt[\nu]{X}} = (x^{1/\theta})^{1/\nu} = x^{1/\theta\nu} = \sqrt[n]{m} \] as we wished to show

**EXAMPLE 4 Using rational exponents to combine radicals**

Consider the expression \( \sqrt[\theta]{\sqrt[\nu]{X^2}} \), where \( x \) and \( y \) are positive.

(a) Rewrite the expression using rational exponents.

(b) Rewrite the expression using only one radical sign.

**SOLUTION**

(a) \( \sqrt[\theta]{\sqrt[\nu]{X^2}} = x^{1/2}y^{2/3} \)

(b) rewriting the fractions using a common denominator.

\( \sqrt[\theta]{\sqrt[\nu]{X^2}} = x^{1/2}y^{2/3} \)

\( = x^{3/6}y^{4/6} \)

\( = \sqrt[\theta]{x^{3/6}y^{4/6}} = \sqrt[\theta]{x^3y^4} \)

**EXAMPLE 5 Using rational exponents to simplify a radical expression**

Rewrite the following expression using rational exponents (assume that \( x \), \( y \), and \( z \) are positive):

\( \sqrt[\theta]{x^2 \sqrt[\nu]{y^3}} \)

**SOLUTION**

\( \sqrt[\theta]{x^2 \sqrt[\nu]{y^3}} = \left( \frac{\sqrt[\theta]{x^2} \sqrt[\nu]{y^3}}{\sqrt[\nu]{z^2}} \right)^{1/2} = \left( \frac{x^{3/2}y^{3/4}}{z^{2/5}} \right)^{1/2} \)

\( = x^{1/2}y^{3/8}z^{-2/8} \)

**EXERCISE SET B.3**

A

**Exercises 1–14 are warm-up exercises to help you become familiar with the definition \( b^{m/n} = (\sqrt[n]{b})^m = \sqrt[n]{b^m} \).** In Exercises 1–8, write the expression in the two equivalent forms \( (\sqrt[n]{b})^m \) and \( \sqrt[n]{b^m} \). In Exercises 9–14, write the expression in the form \( b^{m/n} \).

1. \( a^{3/5} \)
2. \( x^{3/7} \)
3. \( 5^{2/3} \)
4. \( 10^{4/5} \)
5. \( (x^2 + 1)^{3/4} \)
6. \( (a + b)^{7/10} \)
7. \( 2^{5/3} \)
8. \( 2^{m+1/3} \)
9. \( \sqrt[5]{2} \)
10. \( \sqrt[4]{R^2} \)
11. \( \sqrt[3]{(1 + u)^2} \)
12. \( \sqrt[2]{(1 + u)^3} \)
13. \( \sqrt[3]{a^2 + b^3} \)
14. \( \sqrt[2]{(a^2 + b^2)^2} \)

In Exercises 15–50, evaluate or simplify each expression. Express the answers using positive exponents. If an expression is undefined (that is, does not represent a real number), say so. Assume that all letters represent positive numbers.

15. \( 16^{1/2} \)
16. \( 100^{1/2} \)
17. \( (1/6)^{1/2} \)
18. \( 0.09^{1/2} \)
19. \( (-16)^{1/2} \)
20. \( (-1)^{1/2} \)
21. \( 625^{1/4} \)
22. \( (1/81)^{1/4} \)
23. \( 8^{3/3} \)
24. \( 0.001^{1/3} \)
25. \( 8^{2/3} \)
26. \( 64^{5/3} \)
27. \( (-32)^{1/5} \)
28. \( (1/125)^{1/5} \)
29. \( (-1000)^{1/3} \)
30. \( (243)^{1/5} \)
31. \( 49^{-1/2} \)
32. \( 121^{-1/2} \)
33. \( (49)^{-1/2} \)
34. \( (-64)^{-1/3} \)
35. \( 36^{-3/2} \)
B.4 REVIEW OF FACTORING

There are many cases in algebra in which the process of factoring simplifies the work at hand. To factor a polynomial means to write it as a product of two or more nonconstant polynomials. For instance, a factorization of \( x^2 - 9 \) is given by

\[ x^2 - 9 = (x - 3)(x + 3) \]

In this case, \( x - 3 \) and \( x + 3 \) are the factors of \( x^2 - 9 \).

There is one convention that we need to agree on at the outset. If the polynomial or expression that we wish to factor contains only integer coefficients, then the factors (if any) should involve only integer coefficients. For example, according to this convention, we will not consider the following type of factorization in this section:

\[ x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2}) \]

because it involves coefficients that are irrational numbers. (We should point out, however, that factorizations such as this are useful at times, particularly in calculus.) As it happens, \( x^2 - 2 \) is an example of a polynomial that cannot be factored using integer coefficients. In such a case we say that the polynomial is irreducible over the integers.

If the coefficients of a polynomial are rational numbers, then we do allow factors with rational coefficients. For instance, the factorization of \( y^2 - \frac{1}{4} \) over the rational numbers is given by

\[ y^2 - \frac{1}{4} = \left( y - \frac{1}{2} \right) \left( y + \frac{1}{2} \right) \]

We’ll consider five techniques for factoring in this section. These techniques will be applied in the next two sections and throughout the text. In Table 1 we sum-

---

**Example 4** in the text to rewrite each expression using rational exponents and expressing only one radical sign. (Assume that \( x, y, \) and \( z \) are positive.)

For Exercises 51–58, follow Example 4 in the text to rewrite each expression in two ways: (a) using rational exponents and (b) using only one radical sign. (Assume that \( x, y, \) and \( z \) are positive.)

For Exercises 59–66, rewrite each expression using rational exponents rather than radicals. Assume that \( x, y, \) and \( z \) are positive.

**Numerical example showing that the formula is not, in general, valid**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Factorization</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt[3]{x^3} )</td>
<td>( x )</td>
</tr>
<tr>
<td>( \sqrt{y^2} )</td>
<td>( y )</td>
</tr>
<tr>
<td>( \sqrt{x^2} )</td>
<td>( x )</td>
</tr>
<tr>
<td>( \sqrt{x^4} )</td>
<td>( x^2 )</td>
</tr>
<tr>
<td>( \sqrt[3]{x^9} )</td>
<td>( x^3 )</td>
</tr>
</tbody>
</table>

**Errors to avoid**

69. \( (a + b)^{1/2} = a^{1/2} + b^{1/2} \)
70. \( (x^2 + y^2)^{1/2} = x + y \)
71. \( (x + v)^{1/3} = x^{1/3} + v^{1/3} \)
72. \( (p^3 + q^3)^{1/3} = p + q \)
73. \( x^{1/m} = \frac{1}{x^m} \)
74. \( x^{-1/2} = \frac{1}{x^2} \)

---

**Factor** (fak’ ter), n. . . . 2. Math. one of two or more numbers, algebraic expressions, or the like, that when multiplied together produce a given product; . . . vt. 10. Math. to express (a mathematical quantity) as a product of two or more quantities of like kind, as 30 = 2 · 3 · 5, or \( x^2 - y^2 = (x + y)(x - y) \). —The Random House Dictionary of the English Language, 2nd ed. (New York: Random House, 1987)
### TABLE 1 Basic Factoring Techniques

<table>
<thead>
<tr>
<th>Technique</th>
<th>Example or formula</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common factor</td>
<td>$3x^4 + 6x^3 - 12x^2 = 3x^2(x^2 + 2x - 4)$</td>
<td>In any factoring problem, the first step always is to look for the common factor of highest degree.</td>
</tr>
<tr>
<td></td>
<td>$4(x^2 + 1) - x(x^2 + 2) = (x^2 + 1)(4 - x)$</td>
<td></td>
</tr>
<tr>
<td>Difference of squares</td>
<td>$x^2 - a^2 = (x - a)(x + a)$</td>
<td>There is no corresponding formula for a sum of squares; $x^2 + a^2$ is irreducible over the integers.</td>
</tr>
<tr>
<td>Trial and error</td>
<td>$x^2 + 2x - 3 = (x + 3)(x - 1)$</td>
<td>In this example the only possibilities, or trials, are (a) $(x - 3)(x - 1)$ (c) $(x + 3)(x - 1)$</td>
</tr>
<tr>
<td></td>
<td>(b) $(x - 3)(x + 1)$ (d) $(x + 3)(x + 1)$</td>
<td>By inspection or by carrying out the indicated multiplications, we find that only case (c) checks.</td>
</tr>
<tr>
<td>Difference of cubes</td>
<td>$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$</td>
<td></td>
</tr>
<tr>
<td>Sum of cubes</td>
<td>$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$</td>
<td></td>
</tr>
<tr>
<td>Grouping</td>
<td>$x^3 - x^2 + x - 1 = (x^3 - x^2) + (x - 1)$</td>
<td>This is actually an application of the common factor technique.</td>
</tr>
<tr>
<td></td>
<td>$= x^2(x - 1) + (x - 1) \cdot 1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$= (x - 1)(x^2 + 1)$</td>
<td></td>
</tr>
</tbody>
</table>

Marize these techniques. Notice that three of the formulas in the table are just restatements of Special Products formulas. Remember that it is the form or pattern in the formula that is important, not the specific choice of letters.

The idea in factoring is to use one or more of these techniques until each of the factors obtained is irreducible. The examples that follow show how this works in practice.

#### EXAMPLE 1 Factoring using the difference-of-squares technique

Factor:  
(a) $x^2 - 49$;  
(b) $(2a - 3b)^2 - 49$.

**SOLUTION**

(a) $x^2 - 49 = x^2 - 7^2$  
  $= (x - 7)(x + 7)$ difference of squares

(b) Notice that the pattern or form is the same as in part (a): It’s a difference of squares. We have

$$(2a - 3b)^2 - 49 = (2a - 3b)^2 - 7^2$$

$$= [(2a - 3b) - 7][(2a - 3b) + 7]$$ difference of squares

$$= (2a - 3b - 7)(2a - 3b + 7)$$

#### EXAMPLE 2 Factoring using common factor and difference of squares

Factor:  
(a) $2x^3 - 50x$;  
(b) $3x^5 - 3x$.

**SOLUTION**

(a) $2x^3 - 50x = 2x(x^2 - 25)$ common factor

$$= 2x(x - 5)(x + 5)$$ difference of squares
B.4 Review of Factoring

EXAMPLE 3 Factoring by trial and error

Factor: (a) \( x^2 - 4x - 5 \); (b) \( (a + b)^2 - 4(a + b) - 5 \).

SOLUTION
(a) \( x^2 - 4x - 5 = (x - 5)(x + 1) \) trial and error

(b) Note that the form of this expression is

\[
(\quad)^2 - 4(\quad) - 5
\]

This is the same form as the expression in part (a), so we need only replace \( x \) with the quantity \( a + b \) in the solution for part (a). This yields

\[
(a + b)^2 - 4(a + b) - 5 = [(a + b) - 5][(a + b) + 1]
\]

\[
= (a + b - 5)(a + b + 1)
\]

EXAMPLE 4 Factoring by trial and error

Factor: \( 2z^4 + 9z^2 + 4 \).

SOLUTION
We use trial and error to look for a factorization of the form \((2z^2 + \_)(z^2 + \_)\). There are three possibilities:

(i) \( (2z^2 + 2)(z^2 + 2) \)  
(ii) \( (2z^2 + 1)(z^2 + 4) \)  
(iii) \( (2z^2 + 4)(z^2 + 1) \)

Each of these yields the appropriate first term and last term, but (after checking) only possibility (ii) yields \( 9z^2 \) for the middle term. The required factorization is then \( 2z^4 + 9z^2 + 4 = (2z^2 + 1)(z^2 + 4) \).

Question: Why didn’t we consider any possibilities with subtraction signs in place of addition signs?

EXAMPLE 5 Two irreducible expressions

Factor: (a) \( x^2 + 9 \); (b) \( x^2 + 2x + 3 \).

SOLUTION
(a) The expression \( x^2 + 9 \) is irreducible over the integers. (This can be discovered by trial and error.) If the given expression had instead been \( x^2 - 9 \), then it could have been factored as a difference of squares. Sums of squares, however, cannot in general be factored over the integers.

(b) The expression \( x^2 + 2x + 3 \) is irreducible over the integers. (Check this for yourself by trial and error.)
EXAMPLE 6  Factoring using grouping and a special product

Factor: \(x^2 - y^2 + 10x + 25\).

SOLUTION
Familiarity with the special product \((a + b)^2 = a^2 + 2ab + b^2\) suggests that we try grouping the terms this way:

\[
(x^2 + 10x + 25) - y^2
\]

Then we have

\[
x^2 - y^2 + 10x + 25 = (x^2 + 10x + 25) - y^2 = (x + 5)^2 - y^2 = [(x + 5) - y][(x + 5) + y]
\]

difference of squares

\[
= (x - y + 5)(x + y + 5)
\]

EXAMPLE 7  Factoring using grouping

Factor: \(ax + ay^2 + bx + by^2\).

SOLUTION
We factor \(a\) from the first two terms and \(b\) from the second two to obtain

\[
ax + ay^2 + bx + by^2 = a(x + y^2) + b(x + y^2)
\]

We now recognize the quantity \(x + y^2\) as a common expression that can be factored out. We then have

\[
a(x + y^2) + b(x + y^2) = (x + y^2)(a + b)
\]

The required factorization is therefore

\[
ax + ay^2 + bx + by^2 = (x + y^2)(a + b)
\]

Alternatively,

\[
ax + ay^2 + bx + by^2
= (ax + bx) + (ay^2 + by^2)
= x(a + b) + y^2(a + b)
= (a + b)(x + y^2)
\]

As you might wish to check for yourself, this factorization can also be obtained by trial and error.

EXAMPLE 8  Factoring sum and difference of cubes

Factor:  
(a) \(t^3 - 125\);  
(b) \(8 + (a - 2)^3\).

SOLUTION
(a) \(t^3 - 125 = t^3 - 5^3 = (t - 5)(t^2 + 5t + 25)\)  
difference of cubes
\( (b) \quad 8 + (a - 2)^3 = 2^3 + (a - 2)^3 \)
\[ = [2 + (a - 2)][2^2 - 2(a - 2) + (a - 2)^2] \quad \text{sum of cubes} \]
\[ = a(4 - 2a + 4 + a^2 - 4a + 4) \]
\[ = a(a^2 - 6a + 12) \]

As you can check now, the expression \( a^2 - 6a + 12 \) is irreducible over the integers. Therefore the required factorization is \( a(a^2 - 6a + 12) \).

For some calculations (particularly in calculus) it’s helpful to be able to factor an expression involving fractional exponents. For instance, suppose that we want to factor the expression
\[ x(2x - 1)^{-1/2} + (2x - 1)^{3/2} \]

The common expression to factor out here is \( (2x - 1)^{-1/2} \); the technique is to choose the expression with the smaller exponent.

**EXAMPLE 9**  
**Factoring with rational exponents**

Factor: \( x(2x - 1)^{-1/2} + (2x - 1)^{3/2} \).

**SOLUTION**
\[ x(2x - 1)^{-1/2} + (2x - 1)^{3/2} = (2x - 1)^{-1/2}[x + (2x - 1)^{3/2 + 1/2}] \]
\[ = (2x - 1)^{-1/2}[x + (2x - 1)^2] \]
\[ = (2x - 1)^{-1/2}(4x^2 - 3x + 1) \]

We conclude this section with examples of four common errors to avoid in factoring. The first two errors in the box that follows are easy to detect; simply multiplying out the supposed factorizations shows that they do not check. The third error may result from a lack of familiarity with the basic factoring techniques listed on page A-25. The fourth error indicates a misunderstanding of what is required in factoring; the final quantity in a factorization must be expressed as a *product*, not a sum, of terms or expressions.

### ERRORS TO AVOID

<table>
<thead>
<tr>
<th>Error</th>
<th>Correction</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + 6x + 9 = (x - 3)^2 )</td>
<td>( x^2 + 6x + 9 = (x + 3)^2 )</td>
<td>Check the middle term.</td>
</tr>
<tr>
<td>( x^2 + 64 = (x + 8)(x + 8) )</td>
<td>( x^2 + 64 ) is irreducible over the integers.</td>
<td></td>
</tr>
<tr>
<td>( x^3 + 64 ) is irreducible over the integers</td>
<td>( x^3 + 64 = (x + 4)(x^2 - 4x + 16) )</td>
<td>Although a sum of squares is irreducible, a sum of cubes can be factored.</td>
</tr>
<tr>
<td>( x^2 - 2x + 3 ) factors as ( x(x - 2) + 3 )</td>
<td>( x^2 - 2x + 3 ) is irreducible over the integers.</td>
<td>The polynomial ( x^2 - 2x + 3 ) is the <em>sum</em> of the expression ( x(x - 2) ) and the constant 3. By definition, however, the factored form of a polynomial must be a <em>product</em> (of two or more nonconstant polynomials).</td>
</tr>
</tbody>
</table>
**EXERCISE SET B.4**

A

In Exercises 1–66, factor each polynomial or expression. If a polynomial is irreducible, state this. (In Exercises 1–6 the factoring techniques are specified.)

1. (Common factor and difference of squares)
   (a) $x^2 - 64$
   (b) $7x^4 + 14x^2$
   (c) $121z^2 - z^2$
   (d) $a^2b^2 - c^2$

2. (Common factor and difference of squares)
   (a) $1 - t^4$
   (b) $x^4 + x^3 + x^2$
   (c) $u^2v^2 - 225$
   (d) $81x^4 - x^2$

3. (Trial and error)
   (a) $x^2 + 2x - 3$
   (b) $x^2 + 3x - 2$
   (c) $x^2 - 2x + 3$

4. (Trial and error)
   (a) $2x^2 - 7x - 4$
   (b) $2x^2 + 7x - 4$
   (c) $-2x^2 + 7x - 4$

5. (Sum and difference of cubes)
   (a) $x^3 + 1$
   (b) $x^3 + 216$
   (c) $1000 - 8x^6$
   (d) $64a^3 - 125$

6. (Grouping)
   (a) $x^3 - 2x^2 + 3x - 6$
   (b) $a^3x + bx - a^3z - bx$
   (c) $144 - x^2$
   (d) $4a^2b^2 + 9c^2$
   (e) $4a^2b^2 - 9(ab + c)^2$

7. (a) $h^5 - h^3$
    (b) $100h^3 - h^3$
    (c) $100(h + 1)^3 - (h + 1)^5$

8. (a) $x^2 - x^2$
    (b) $3x^4 - 48x^2$
    (c) $3(x + h)^2 - 48(x + h)^2$

9. (a) $x^2 - 13x + 40$
    (b) $x^2 - 13x + 40$
    (c) $x^2 - 3x + 36$
    (b) $x^2 - 3x + 36$
    (c) $3x^2 - 22x - 16$
    (b) $3x^2 - x - 16$
    (c) $6x^2 + 13x - 5$
    (b) $6x^2 + 13x - 5$
    (c) $6x^2 - x - 5$

10. (a) $t^4 + 2t^2 + 1$
    (b) $t^4 - 2t^2 + 1$
    (b) $t^4 - 2t^2 + 1$
    (b) $t^4 - 2t^2 + 1$
    (b) $t^4 - 2t^2 + 1$
    (b) $4x^3 + 20x^2 - 25x$
    (b) $4x^3 - 20x^2 + 25x$

11. (a) $ab - bc + a^2 - ac$
    (b) $(u + v)x - xy + (u + v)^2 - (u + v)y$
    (c) $a(x + 5)^2 + b(x + 5)^2$

12. (a) $3(x + 5)^2 + 2(x + 5)^2$
    (b) $3(x + 5)^2 + 2(x + 5)^2$
    (b) $x^2z + x2t + xzy + yt$
    (b) $x^2z + x2t + xzy + yt$

13. (a) $27 - (a - b)^3$
    (b) $(a + b)^3 - 8c^3$
    (c) $8a^3 + 27b^3 + 2a + 3b$
    (b) $p^4 + 4$
    (b) $p^4 - 4$
    (b) $p^4 - 4$

14. (a) $x^3 - y^3 + x - y$
    (b) $p^4 - 1$
    (b) $p^8 - 1$

15. $x^2 + 3x^2 + 3x + 1$
    (b) $x^2 + 16y^2$
    (c) $25 - c^2$

16. $25 - 81$
    (b) $4 - y^2$
    (c) $4 - y^2$

17. $a^4 - (b + c)^4$
    (b) $3x^2 - 2x^2 + 3x + 1$
    (b) $x^2 + 16y^2$

18. 26. 47. 50.
    27. 51. 52.

19. $125m^3 - 1$
   (b) $\sqrt{x^2 + xy + y^2}$
   (b) $64(x - a)^3 - x + a$

20. $x^2 - a^2 + y^2 - 2xy$
   (b) $21x^3 + 82x^2 - 39x$
   (b) $564(x - a)^3 - x + a$

21. $21x^3 + 82x^2 - 39x$
   (b) $64(x - a)^3 - x + a$
   (b) $21x^3 + 82x^2 - 39x$

22. $x^2a^2 - 8y^2a^2 - 4x^2b^2 + 32y^2b^2$
   (b) $12x^2 + 25 - 4x^2 - 9y^2$

23. $ax^2 + (1 + ab)xy + by^2$
   (b) $ax^2 + (a + b)x + b$
   (b) $ax^2 + (a + b)x + b$

24. $5a^2 - 11a + 10)^2 - 4a^2 - 15a + 6)^2$
   (b) $(x + 1)\frac{1}{2} - (x + 1)^{3/2}$
   (b) $(x + 1)^3/2 + (x^2 + 1)^{3/2}$

25. $(x + 1)^{3/2} - (x + 1)^{-3/2}$
   (b) $(x + 1)^{3/2} + (x^2 + 1)^{-3/2}$
   (b) $(x + 1)^{3/2} - (x + 1)^{3/2}$

26. $(2x + 3)^{1/2} - (2x + 3)^{1/2}$
   (b) $(2x + 3)^{1/2} - (2x + 3)^{1/2}$
   (b) $(ax + b)^{-1/2} - \sqrt{ax + b}/b$

In Exercises 77–88, evaluate the given expressions using factoring techniques. (The point here is to do as little actual arithmetic as possible.)

77. (a) $100^2 - 99^2$
    (b) $8^3 - 6^3$
    (c) $1000^2 - 999^2$

78. (a) $10^3 - 9^3$
    (b) $50^3 - 49^3$
    (c) $15^3 - 10^3$

79. $y^3 - (p + q)y^2 + (p^2q + pq^2)y - p^2q^2$
80. (a) Factor $x^4 + 2x^3y^2 + y^4$. Hint: Add and subtract a term. [Keep part (a) in mind.]

B

In Exercises 69–73, factor each expression.

69. $A^3 + B^3 + 3AB(A + B)$
70. $p^3 - q^3 - p(p^2 - q^2) + q(p - q)^2$
71. $2tx(a^2 + x^2)^{1/2} - x(ta^2 + x^2)^{1/2}$
72. $\frac{1}{2}(x - a)^{-1/2}(x + a)^{1/2} - \frac{1}{2}(x + a)^{1/2}(x - a)^{-1/2}$
73. $y^3 - (p + q)y^2 + (p^2q + pq^2)y - p^2q^2$
74. (a) Factor $x^4 + 2x^3y^2 + y^4$. Hint: Add and subtract a term. [Keep part (a) in mind.]

(f) Factor $x^4 - y^4$ as a difference of squares.

(d) Factor $x^6 - y^6$ as a difference of cubes. Use the result in part (b) to obtain the same answer as in part (c).]
The rules of basic arithmetic that you learned for working with simple fractions are also used for fractions involving algebraic expressions. In algebra, as in arithmetic, we say that a fraction is reduced to lowest terms or simplified when the numerator and denominator contain no common factors (other than 1 and -1). The factoring techniques that we reviewed in Appendix B.4 are used to reduce fractions. For example, to reduce the fraction \( \frac{x^2 - 9}{x^2 + 3x} \), we write

\[
\frac{x^2 - 9}{x^2 + 3x} = \frac{(x - 3)(x + 3)}{x(x + 3)} = \frac{x - 3}{x}
\]

In the box that follows, we review two properties of negatives and fractions that will be useful in this section. Notice that Property 2 follows from Property 1 because

\[
\frac{a - b}{b - a} = \frac{a - b}{-(a - b)} = -1
\]

**PROPERTY SUMMARY** Negatives and Fractions

<table>
<thead>
<tr>
<th>Property</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( b - a = -(a - b) )</td>
<td>( 2 - x = -(x - 2) )</td>
</tr>
<tr>
<td>2. ( \frac{a - b}{b - a} = -1 )</td>
<td>( \frac{2x - 5}{5 - 2x} = -1 )</td>
</tr>
</tbody>
</table>

**EXAMPLE 1** Using Property 2 to simplify a fractional expression

Simplify: \( \frac{4 - 3x}{15x - 20} \)

**SOLUTION**

\[
\frac{4 - 3x}{15x - 20} = \frac{4 - 3x}{5(3x - 4)} = \frac{1}{5} \left( \frac{4 - 3x}{3x - 4} \right) = \frac{-1}{5} \text{ using Property 2}
\]

In Example 2 we display two more instances in which factoring is used to reduce a fraction. After that, Example 3 indicates how these skills are used to multiply and divide fractional expressions.

**EXAMPLE 2** Using factoring to simplify a fractional expression

Simplify: (a) \( \frac{x^3 - 8}{x^2 - 2x} \), (b) \( \frac{x^2 - 6x + 8}{a(x - 2) + b(x - 2)} \)
Appendix B

SOLUTION

(a) \[ \frac{x^2 - 8}{x^2 - 2x} = \frac{(x-2)(x^2 + 2x + 4)}{x(x-2)} = \frac{x^2 + 2x + 4}{x} \]

(b) \[ \frac{x^2 - 6x + 8}{a(x-2) + b(x-2)} = \frac{(x-2)(x-4)}{(x-2)(a+b)} = \frac{x-4}{a+b} \]

EXAMPLE 3 Using factoring to simplify products and quotients

Carry out the indicated operations and simplify:

(a) \[ \frac{x^3}{2x^2 + 3x} \cdot \frac{12x + 18}{4x^2 - 6x} = \frac{x^3}{2x(2x - 3)} \cdot \frac{6(2x + 3)}{2x(2x - 3)} = \frac{3x}{2x - 3} \]

(b) \[ \frac{2x^2 - x - 6}{x^2 + x + 1} \cdot \frac{x^3 - 1}{4 - x^2} = \frac{(2x + 3)(x - 2)}{(x^2 + x + 1)} \cdot \frac{(x - 1)(x^2 + x + 1)}{(2 - x)(2 + x)} = \frac{2 + x}{2 + x}, \quad \text{using the fact that} \quad \frac{x - 2}{2 - x} = -1 \]

(c) \[ \frac{x^2 - 144}{x^2 - 4} + \frac{x + 12}{x + 2} = \frac{(x - 12)(x + 12)}{(x - 2)(x + 2)} \cdot \frac{x + 2}{x + 12} = \frac{x - 12}{x - 2} \]

As in arithmetic, to combine two fractions by addition or subtraction, the fractions must have a common denominator, that is, the denominators must be the same. The rules in this case are

\[ \frac{a}{b} + \frac{c}{d} = \frac{a + c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{d} = \frac{a - c}{b} \]

For example, we have

\[ \frac{4x - 1}{x + 1} - \frac{2x - 1}{x + 1} = \frac{4x - 1 - (2x - 1)}{x + 1} = \frac{4x - 1 - 2x + 1}{x + 1} = \frac{2x}{x + 1} \]

Fractions with unlike denominators are added or subtracted by first converting to a common denominator. For instance, to add \( \frac{9}{a} \) and \( \frac{10}{a^2} \), we write

\[ \frac{9}{a} + \frac{10}{a^2} = \frac{9}{a} \cdot \frac{a}{a} + \frac{10}{a^2} = \frac{9a}{a^2} + \frac{10}{a^2} = \frac{9a + 10}{a^2} \]
Notice that the common denominator used was \( a^2 \). This is the **least common denominator**. In fact, other common denominators (such as \( a^3 \) or \( a^5 \)) could be used here, but that would be less efficient. In general, the least common denominator for a given group of fractions is chosen as follows. Write down a product involving the irreducible factors from each denominator. The power of each factor should be equal to (but not greater than) the highest power of that factor appearing in any of the individual denominators. For example, the least common denominator for the two fractions \( \frac{1}{(x + 1)^2} \) and \( \frac{1}{(x + 1)(x + 2)} \) is \((x + 1)^2(x + 2)\). In the example that follows, notice that the denominators must be in factored form before the least common denominator can be determined.

**EXAMPLE 4 Using the least common denominator**

Combine into a single fraction and simplify:

(a) \( \frac{3}{4x} + \frac{7x}{10y^2} - 2 \);

(b) \( \frac{x}{x^2 - 9} - \frac{1}{x + 3} \);

(c) \( \frac{15}{x^2 + x - 6} + \frac{x + 1}{2 - x} \)

**SOLUTION**

(a) Denominators: \( 2^2 \cdot x; 2 \cdot 5 \cdot y^2; 1 \)

Least common denominator: \( 2^2 \cdot 5 \cdot x \cdot y^2 = 20xy^2 \)

\[
\frac{3}{4x} + \frac{7x}{10y^2} - \frac{2}{1} = \frac{3 \cdot 5y^2}{4x \cdot 5y^2} + \frac{7x \cdot 2y}{10y^2 \cdot 2x} - \frac{2 \cdot 10xy^2}{1 \cdot 10xy^2} = \frac{15y^2 + 14x^2 - 40xy^2}{20xy^2}
\]

*Note:* The final numerator is irreducible over the integers.

(b) Denominators: \( x^2 - 9 = (x - 3)(x + 3); x + 3 \)

Least common denominator: \( (x - 3)(x + 3) \)

\[
\frac{x}{x^2 - 9} - \frac{1}{x + 3} = \frac{x}{(x - 3)(x + 3)} - \frac{1}{x + 3} = \frac{x}{x + 3} - \frac{(x - 3)}{x + 3} = \frac{x - (x - 3)}{x + 3} = \frac{3}{x + 3}
\]

Note in the last line that we were able to simplify the answer by factoring the numerator and reducing the fraction. (This is why we prefer to leave the least common denominator in factored form, rather than multiplying it out, in this type of problem.)
The next four examples illustrate using the least common denominator to simplify compound fractions, or complex fractions, which are fractions “within” fractions.

**EXAMPLE 5** Simplifying a compound fraction

\[
\frac{1}{3a} - \frac{1}{4b}
\]

Simplify: \( \frac{5}{6a^2} + \frac{1}{b} \).

**SOLUTION**

The least common denominator for the four individual denominators 3a, 4b, 6a², and b is 12a²b. Multiplying the given expression by \( \frac{12a^2b}{12a^2b} \), which equals 1, yields

\[
\frac{12a^2b}{12a^2b} \cdot \frac{\frac{1}{3a} - \frac{1}{4b}}{\frac{5}{6a^2} + \frac{1}{b}} = \frac{4ab - 3a^2}{10b + 12a^2}
\]

**EXAMPLE 6** Simplifying a fraction containing negative exponents

Simplify: \((x^{-1} + y^{-1})^{-1}\). (The answer is not \(x + y\).)

**SOLUTION**

After applying the definition of negative exponent to rewrite the given expression, we’ll use the method shown in Example 5.

\[
(x^{-1} + y^{-1})^{-1} = \left(\frac{1}{x} + \frac{1}{y}\right)^{-1} = \frac{1}{\frac{1}{x} + \frac{1}{y}}
\]

\[
= \frac{xy}{1} \cdot \frac{1}{x + y} = \frac{xy}{y + x}
\]

**EXAMPLE 7** Simplifying a compound fraction

\[
\frac{1}{x + h} - \frac{1}{x}
\]

Simplify: \( \frac{1}{x + h} - \frac{1}{x} \). (This type of expression occurs in calculus.)

**SOLUTION**

\[
\frac{1}{x + h} - \frac{1}{x} = \frac{(x + h)x}{(x + h)x} \cdot \frac{1}{x + h} - \frac{1}{x}
\]

\[
= \frac{x}{(x + h)x} - \frac{1}{(x + h)x} = \frac{x - (x + h)}{(x + h)x} = -\frac{h}{(x + h)xh} = -\frac{1}{x(x + h)}
\]
EXAMPLE 8 Simplifying a compound fraction

\[ \frac{x - \frac{1}{x^2}}{x - \frac{1}{x^2}} \]

Simplify: \( \frac{1}{x^2 - 1} \)

**SOLUTION**

\[
\frac{x - \frac{1}{x^2}}{x^2 - 1} = \frac{x - \frac{1}{x^2}}{\frac{1}{x^2}} \cdot \frac{1}{x^2 - 1} = \frac{x^2 - 1}{x^2} = \frac{1}{x^2 - 1} = \frac{x - 1}{x^2 - 1} = \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)} = \frac{x^2 + x + 1}{x + 1}
\]

**EXERCISE SET B.5**

**A**

In Exercises 1–12, reduce the fractions to lowest terms.

1. \( \frac{x^2 - 9}{x + 3} \)
2. \( \frac{25 - x^3}{x - 5} \)
3. \( \frac{x^2 + 2x + 4}{x^2 - 16} \)
4. \( \frac{x^2 - x - 20}{2x^2 + 7x - 4} \)
5. \( \frac{9ab - 12b^2}{6a^2 - 8ab} \)
6. \( \frac{a^2 + b^2}{ax^2 + bx^2} \)
7. \( \frac{a^3 + a^2 + a + 1}{a^2 - 1} \)
8. \( \frac{x^3 - y^3}{(x - y)^3} \)
9. \( \frac{x^4 - y^4}{(x^4 + x^2y^2 + xy^4)(x - y)^2} \)

In Exercises 13–55, carry out the indicated operations and simplify where possible.

13. \( \frac{2}{x - 2} \cdot \frac{x^2 - 4}{x + 2} \)
14. \( \frac{ax + 3}{2a + 1} + \frac{a^2x^2 + 3ax}{4a^2 - 1} \)
15. \( \frac{x^2 - x - 2}{x^2 + x - 12} \cdot \frac{x^2 - 3x}{x^2 - 4x + 4} \)
16. \( \frac{3x^2 + 2tx - x^2}{3t^2 + 2tx - x^2} \)
17. \( \frac{x^3 + y^3}{x^3 - 4xy + 3y^2} + \frac{x^3 - y^3}{x^2 - 2xy - 3y^2} \)
18. \( \frac{a^2 - a - 42}{a^2 + 216a} + \frac{a^2 - 49}{a^3 - 6a^2 + 36a} \)
19. \( \frac{4}{x - \frac{2}{x^2}} \)
20. \( \frac{1}{\frac{3x}{x^2} + \frac{1}{5x^2} - \frac{1}{30x^3}} \)
21. \( \frac{\frac{3}{a} + \frac{1}{b}}{\frac{1}{a}} + \frac{1}{c} \)
22. \( \frac{\frac{4}{x - 4}}{\frac{4}{x + 1}} \)
23. \( \frac{1}{\frac{1}{x} + \frac{1}{x^2}} \)
24. \( \frac{1}{\frac{1}{x} - \frac{1}{x^2}} \)
25. \( \frac{a}{x - 1} + \frac{2ax}{(x - 1)^2} + \frac{3ax^2}{(x - 1)^2} \)
26. \( \frac{a^2 + 5a - 4}{a^2 - 16} - \frac{2a}{2a^2 + 8a} \)
27. \( \frac{4}{x - 5} - \frac{4}{5 - x} \)
28. \( \frac{x^2 + 3}{2x + 2} - \frac{5}{x^2 - 1} + \frac{1}{x + 1} \)
29. \( \frac{1}{x^2 + x - 20} - \frac{1}{x^2 - 8x + 16} \)
30. \( \frac{6x^2 + 5x - 4}{2x + p} + \frac{1}{3x^2 + 4x - \frac{1}{2x - 1}} \)
31. \( \frac{2p^2 - 9pq - 5q^2}{p^2 - 5pq} \)
32. \( \frac{1}{x - 1} + \frac{1}{x^2 - 1} + \frac{1}{x^2 - 1} \)
Consider the three fractions 

(a) If \( a = 1, b = 2, \) and \( c = 3, \) find the sum of the three fractions. Also compute their product. What do you observe?  

(b) Show that the sum and the product of the three given fractions are, in fact, always equal.