3.4.10 Logarithmic differentiation

Textbook pages 231-233

In the previous examples, we saw that taking the derivative of the logarithm of a complicated function involving products and powers is actually very easy. By comparison, taking the derivative of the function itself could be more complicated. Based upon this idea, we now introduce the concept of *logarithmic differfentiation*. The following example will illustrate how powerful this method is.

EXAMPLE: Find the derivative of

$$f(x) = \frac{e^{2x}(2x-1)^6}{(x+5)^3(4-7x)}$$

IDEA:

- Take the logarithm of both sides of this equality, and expand the log.
- Take the derivative of both sides of this equality
- Multiply both sides by $f(x)$ to obtain $f'(x)$

Example: Find the derivative of

$$f(x) = \frac{2^x(x+1)^2}{(x^2+2x-1)^3(2+7x)}$$

Example: The logarithmic differentiation is also useful to find derivatives of complicated functions such as $f(x) = x^x$.

Proof of the power rule in the general case: Earlier in the lecture, we proved the power rule

for positive integer vales of α. We are now able to prove it in the general case, using logarithmic differentiation.

3.4.11 Implicit differentiation

Textbook pages 200-203

So far, we have only considered taking derivatives of functions defined explicitly as $y = f(x)$. In that case, y is a genuine, uniquely defined function of x, and the derivative $f'(x)$ is the slope of the tangent to the graph $y = f(x)$ at every point.

However, in many cases one may also like to know the tangent to curves which are not defined as functions of x. For example, in the case of a circle,

In this case, x and y are related by the above equation, or in other words, y is defined *implicitly* in terms of x through this equation.

QUESTION: What is the tangent to the circle $x^2 + y^2 = 1$ at the point $(\sqrt{2}, \sqrt{2})$?

How can we find the equation for the tangent in this case? We can consider two methods: the direct method, which is more complicated (and sometimes impossible) , and the method of implicit differentiation, which is easier (and always possible).

DIRECT METHOD: If y is not defined in terms of x, the first thing to do is to solve for y. In the case of the circle,

Then take the derivative of $y(x)$:

Once we have the derivative, we have the slope of the tangent, so we can calculate the line equation with the point-slope formula:

PROBLEM: If we had a more complicated equation relating x and y, it would be impossible to solve for y, so what could we do in this case?

IDEA: Although technically we cannot write $y = f(x)$ (because the circle is not the graph of a function), we will still do it and then take the derivative of the *whole equation*:

- Step 1: Take the equation and substitute $f(x)$ for y.
- Step 2: Take the derivative of the equation
- Step 3: Substitute back y for $f(x)$
- Step 4: If $f'(x)$ is needed, solve for it.

This gives the slope of the curve at any point (x, y). Then we use the point given $(\sqrt{2}, \sqrt{2})$ to get the slope at the point desired.

INTEREST OF THE METHOD: This method can be used for any complicated curve! A famous curve is called the Bifolium, and is given by the equation

$$(x^2 + y^2)^2 = 4x^2 y$$

Question 1: Check that the point A$\left(\sqrt{\frac{3}{4}+\frac{1}{\sqrt{2}}}, \frac{1}{2}\right)$ is on the Bifolium.

Question 2: What is the slope of the tangent to the Bifolium at any point (x, y)?

Question 3: What is the equation of the tangent to the Bifolium at the point A?

Question 4: At which points does the Bifolium have a horizontal tangent?

Check your understanding of Lecture 13

- **Logarithmic differentiation:**
  - Let $f(x) = g(x)^\alpha$. Using the chain rule, prove that $f'(x) = \alpha g'(x)g(x)^{\alpha-1}$. Prove the same result using logarithmic differentiation (i.e. say $\ln(f(x)) = \ln(g(x)^\alpha)$ and take the derivative on both sides).
  - Textbook problems page 234 number 63, 65, 67, 71, 73, 75.
- **Tangents to the circle**
  Consider the equation $x^2 + y^2 = a^2$. What geometrical object does this correspond to (give a precise description)? Find the slope of the tangent to this curve at any point on the curve. If the point with $x-$coordinate c is on the curve, what is its y-coordinate? Deduce the equation for the tangent to this curve at any point on it.
- **Tangents to an ellipse**
  Consider the ellipse $x^2+2y^2 = 4$. What is the y-coordinate of a point with $x-$coordinate 1? Find the equation of the tangent to this ellipse at this point.
- Using the implicit differentiation method, find the slope of the tangents to each of these curves at any point (x, y) on the curve:
  - $x^4 + y^4 = 2$
  - $x^2y + 3e^y = \ln(x)$
  - $\sin(x) = \cos(y)$
  - $3xy + 4y^6 \sin(x) = y \sin(y)$

3.5 Higher order derivatives

3.5.1 Higher order derivatives

Higher order derivatives are simply successive derivatives of the same function.

Second order derivative:

Third order derivative:

n-th order derivative:

Examples:

- $f(x) = (2x+1)^3$
  - $f'(x) =$
  - $f''(x) =$
  - $f^{(3)}(x) =$
  - $f^{(4)}(x) =$
  - $f^{(n)}(x) =$
- $f(x) = e^x$
  - $f'(x) =$
  - $f''(x) =$

- $f^{(3)}(x) =$

- $f^{(n)}(x) =$

3.5.2 Graphical interpretation of the second-order derivative

The second derivative is the derivative of the derivative. Therefore

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EXAMPLE 1: $f(x) = x^2$

EXAMPLE 2: $f(x) = x^3$

SO WE NOTE THAT:

DEFINITION:

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3.5.3 Graphing with derivatives (part 3)

We now see that to get as much information as possible about a function, we should study

- its sign, to know whether it is positive or negative

- the sign of its derivative, to know whether it is increasing or decreasing
- the sign of its second derivative, to know whether it has a positive or a negative curvature.

Example 1: $f(x) = x(x-1)(x+1)$

EXAMPLE 2: $f(x) = \frac{x^2}{x^2+k^2}$.

CHECK YOUR UNDERSTANDING OF LECTURE 14

- **Second order derivatives**
  Calculate the second order derivative of the following functions:
  - $\sin(x)$
  - e^{ax} where a is an arbitrary constant
  - e^{-x^2}
  - $x^3 - 2x + 1$
- **Higher order derivatives**
  Calculate the sucessive derivatives (first, second, third, etc..) of
  - $\sin(kx)$ (where k is an arbitrary constant)
  - e^{ax} (where a is an arbitrary constant)

  Can you deduce the general formula for the n-th derivative of these functions?
- **Graphing with derivatives (1)**
  For each of the following, sketch the function $f(x)$ using what you know of basic functions. Then, by looking at the graph, give your best shot at guessing what the derivative $f'(x)$ may look like. Finally, calculate the derivative and plot it with a graphing calculator to check that your guess was correct. NOTE: this exercise is only worth doing if you indeed do it the way suggested.
  - $f(x) = 1/x$
  - $f(x) = x^3 + 1$
- **Graphing with derivatives (2)**
  For each of the following, sketch the function $f'(x)$ using what you know of basic functions. Then, by looking at the graph, give your best shot at guessing what the function $f(x)$ may look like if it has the derivative $f'(x)$, and if $f(0)$ has the value given in the question. Finally, plot the function $f(x)$ as given on a graphic calculator to check whether your guess was correct. NOTE: this exercise is only worth doing if you indeed do it the way suggested.
  - $f'(x) = 3x + 1$ with $f(0) = -2$. Check: $f(x) = \frac{3x^2}{2} + x - 2$
  - $f'(x) = x^2 - 1$ with $f(0) = 1$. Check: $f(x) = \frac{x^3}{3} - x + 1$
- **Graphing with derivatives (3)**
  For one (or more) of the following functions perform the steps:
  - Determine the parity of the function (odd, even, neither)
  - Find the roots and asymptotes of the function
  - Find the limits of the function at $+\infty$ and $-\infty$

- Calculate $f'(x)$, and identify the roots of $f'(x)$. Using a signs table, or otherwise, find all the stationary points, and identify the intervals where the function is increasing/decreasing
- Calculate $f''(x)$ and identify the roots of $f''(x)$. Using a signs table, or otherwise, find all the inflection points, and identify the intervals where the function is concave-up/down.
- Using all of the above information, sketch the function $f(x)$.

The functions to work on are the following

- $f(x) = x^3 - x^2$ (note: you will have to factor it to find the roots)
- $g(x) = \frac{x}{x+1}$ (cf. Textbook Problem page 270 number 14)

3.6 Summary of the properties of graphs and functions; a bit of vocabulary

What can we learn from the derivative function $f'(x)$ about the graph $y = f(x)$?

- the derivative of $f(x)$ is at all point equal to the slope of the graph $y = f(x)$.
- if $f'(x) > 0$, $f(x)$ is a *monotonically increasing* function.
- if $f'(x) < 0$, $f(x)$ is a *monotonically decreasing* function.
- the points where $f'(x) = 0$ are *stationary points*, and either correspond to local minima, local maxima or simply points which have a horizontal slope.

What can we learn from the second derivative function $f''(x)$ about the graph $y = f(x)$?

- the second derivative of $f(x)$ is at all point equal to the curvature of the graph $y = f(x)$.
- if $f''(x) > 0$, the graph has a *positive curvature*, it is concave up.
- if $f''(x) < 0$, the graph has a *negative curvature*, it is concave down
- if $f''(x) = 0$, the graph has an *inflection point* (where the concavity changes from up to down).

Using the two results above, we can say that:

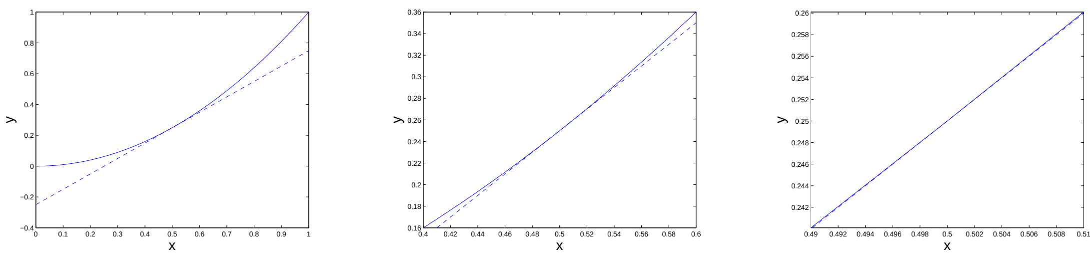
- If the function has a point where $f'(x) = 0$ and $f''(x) > 0$ then it is a local minimum
- If the function has a point where $f'(x) = 0$ and $f''(x) < 0$ then it is a local maximum
- If the function has a point where $f'(x) = 0 = f''(x)$ then the point is an inflection point with horizontal slope

3.7 Local linearity

Textbook page 235

Given a function $f(x)$, the tangent to the curve $y = f(x)$ at the point with $x-$coordinate $x = c$ is given by the equation

In the Figure below, we see that near the point of contact, the graphs of the function (solid line) and of the tangent (dashed line) are very close to each other, and the closer to the point, the closer they are.



(a) (b) (c)

Local linearization: Since the graph of $y = f(x)$ is close to the graph of $y = f(c) + f'(c)(x - c)$ near the point $(c, f(c))$, we write that

In other words, the expression $f(c) + f'(c)(x - c)$ is an *approximation* to $f(x)$ near the point $x = c$.

Alternative form:

Alternative derivation of the local approximation:

Notes:

-
-

Examples:

- What is the local linearization of $\sin(x)$ near $x = 0$?
- What is the local linearization of e^x near $x = 0$?
- What is the local linearization of x^2 near $x = 1$?
- What is the local linearization of $\tan(x)$ near $x = \pi/4$?

3.8 L'Hospital's rule: limits revisited

Textbook pages 299-307

L'Hospital's rule is a very powerful tool for finding limits of indeterminate limits such as

Rule for the indeterminate form $\frac{0}{0}$:

Examples for the indeterminate form $\frac{0}{0}$:

- Find the limit of $\frac{\sin(x)}{x}$ at $x = 0$.

- Find the limit of $\frac{x^2-9}{x-3}$ at $x = 3$.

- Find the limit of $\frac{\tan(x)}{x}$ at $x = 0$.

PROOF FOR THE INDETERMINATE FORM $\frac{0}{0}$:

RULE FOR THE INDETERMINATE FORM $\frac{\infty}{\infty}$:

EXAMPLES FOR THE INDETERMINATE FORM $\frac{\infty}{\infty}$:

- Find $\lim_{x\to+\infty} \frac{\ln(x)}{x}$.

- Find $\lim_{x \to (\pi/2)-} \frac{\tan(x)}{1+\tan(x)}$.

The proof of this indeterminate form is more difficult, so we will simply assume that it is true.

ADVANTAGES:

- L'Hospital Rule can be applied multiple times

- L'Hospital Rule can also be used to find limits of other indeterminate forms provided we can cast them into either of the above 0/0 or ∞/∞.

EXAMPLES:

- Find $\lim_{x \to 0} \frac{x^2}{\sin^2(x)}$.

- Find $\lim_{x \to 0} \left(\frac{1}{\sin(x)} - \cot(x) \right)$

Important note: L'Hospital rule ONLY works for these two indeterminate forms. Do not try to use it without verifying that the limit is one of those forms first. For example, you cannot use L'Hospital Rule on the following problems:

- $\lim_{x\to 3} \frac{x^2-1}{x-3}$

- $\lim_{x\to 0} \frac{x}{\ln(x)}$

- $\lim_{x\to \infty} \frac{e^{-x}}{x^2}$

(and so forth)...

CHECK YOUR UNDERSTANDING OF LECTURE 15

- **Local linearity:**
  Using the concept of local linearity, find linear approximations to the following functions near the point $x = 1$:
  - $f(x) = \sqrt{x}$
  - $f(x) = \ln(x)$
  - $f(x) = 3e^{x^2}$
- **Limits using L'Hospital's Rule**
  Do as many problems as you can from the following list: Textbook page 307 numbers 1 through 32 (a good start would be all of the odd-numbered, or all of the even numbered problems on that list).
- **Further examples of L'Hospital's Rule**
  Textbook page 308 number 51, 56.
