F.T. of Calc: $\frac{d}{dx}\left(\int_a^x f(x)\,dx\right) = f(x)$

Properties of Random Variables

G.R., September 22, 2006

#	D.R.V. Prop.	Description
1	PMF	$p_X(x) = P(X = x)$
2	PMF range	$0 \le p_X(x_k) \le 1$
3	PMF range	$p_X(x) = 0$ if x is not in the sample space of X.
4	PMF sums to 1	$\sum_k p_X(x_k) = 1$
5	discrete CDF	$F_X(x) = P(X \le x) = \sum_{x_k \le x} p_X(x_k)$
6	mean	$\mu_X = E(X) = \sum_k x_k p_X(x_k)$
7	nth-moment	$E(X^n) = \sum_k x_k^n p_X(x_k)$
8	variance	$\sigma_X^2 = \text{Var}(X) = E([X - E(X)]^2)$
9		$\sigma_X^2 = E(X^2) - [E(X)]^2$
10		$\sigma_X^2 = \sum_k (x_k - \mu_X)^2 p_X(x_k)$
11	skew	$\alpha_3 = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{\mu_3}{\sigma^3}$
12	kurtosis	$\alpha_4 = \frac{E[(X-\mu)^4]}{\sigma^4} = \frac{\mu_4}{\sigma^4}$
13	Joint CDF	$F_{XY}(x,y) = P(X \le x, Y \le y)$ $P(A) = F(x), P(B) = F(y)$, $F_{XY}(x,y) = P(A \bigcap B)$
14	independence	If A and B are independent events on S, $F(x,y) = P(A \bigcap B) = P(A)P(B)$ $F_{XY}(x,y) = F_X(x)F_Y(y)$
15		Two r.v.'s X and Y are independent if $F_{XY}(x,y) = F_X(x)F_Y(y) \;\; \forall x, y.$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

TABLE I

SUMMARY OF PROPERTIES OF **DISCRETE RANDOM VARIABLES**.

Formal test for independence

1. Form joint distribution $f_{xy} = f_x \circledast f_y$
2. Identify f_x, f_y individually
3. Does $f_{xy} = f_x \cdot f_y$?
   If so, independent

#	C.R.V. Prop.	Description
1	PDF	$f_X(x)$ is piecewise continuous
2	PDF	$f_X(x) \ge 0$
3	PDF integrates to 1	$\int_{-\infty}^{\infty} f_X(x)dx = 1$
4	Prob. from PDF	$P(a < X \le b) = \int_a^b f_X(x)dx$
5	continuous CDF	$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(s)ds$
6	$P(a < X \le b) = P(a \le X \le b) = P(a \le X < b) = P(a < X < b)$	
7	Prob. from CDF	$\int_a^b f_X(x)dx = F_X(b) - F_X(a)$
8	mean	$\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$
9	nth moment	$E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)dx$
10	variance	$\sigma_X^2 = \text{Var}(X) = E([X - E(X)]^2)$
11		$\sigma_X^2 = E(X^2) - [E(X)]^2$
12		$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)dx$
13	skew	$\alpha_3 = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{\mu_3}{\sigma^3}$
14	kurtosis	$\alpha_4 = \frac{E[(X-\mu)^4]}{\sigma^4} = \frac{\mu_4}{\sigma^4}$
14	joint pdf	$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$
15	joint CDF	$F_{XY} = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)dxdy$
16	pdfs	$f(x,y) \ge 0$
17		$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$
18		$P[(X,Y) \in A] = \int\int_{R_A} f(x,y)dxdy$
19		$P(a < X \le b, c < Y \le d) = \int_c^d \int_a^b f(x,y)dxdy$
20		$F_X(x) = F_{XY}(x,\infty) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy$
21		$f_X(x) = \frac{dF_X(x)}{dx} = \int_{-\infty}^{\infty} f_{XY}(x,\eta)d\eta$
22	(3.30)	$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$
23		Independence: $F_{XY}(x,y) = F_X(x)F_Y(y)$
24		$\frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}F_X(x)\frac{\partial}{\partial y}F_Y(y)$
25		$f_{XY}(x,y) = f_X(x)f_Y(y)$
26		$\mu_X = E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)dxdy$
27		$\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$

TABLE II

p123 SUMMARY OF PROPERTIES OF **CONTINUOUS RANDOM VARIABLES**.

- $Z = g(X,Y)$ defines a new r.v. "Z"
- define D_Z the subset of R_{XY} s.t. $g(x,y) \le z$
- Then $(Z \le z) = [g(X,Y) \le z] = \{(X,Y) \in D_Z\}$
- Then $F_Z(z) = P(Z \le z) = P[g(X,Y) \le z] = P\{(X,Y) \in D_Z\} = \iint_{D_Z} f_{XY}(x,y)\,dx\,dy$

D.R.V: Bernoulli Random Variable

#	Property	Description
1	sample space	$S_X = 0, 1$
2	p_k	$p_0 = q = 1 - p,\ p_1 = p,\ 0 \leq p \leq 1$
3		$E[X] = p$
4		Var $[X] = p \cdot (1-p)$
5	gen. fnc.	$G(z) = (q + pz)$

X is the value of indicator fnc. I_A for some event A

..$X = 1$ if A occurs, and 0 otherwise.

D.R.V: Binomial Random Variable

#	Property	Description
1	S_X	$S_X = 0, 1, ..., n$
2	p_k	$p_k = \binom{n}{k} p^k (1-p)^{n-k},\ k = 0, 1, ..n$
3	$E[X]$	$E[X] = np$
4	Var[X]	Var $[X] = np \cdot (1-p)$
5	$G(z)$	$G(z) = (q + pz)^n$

X is the number of successes in n Bernoulli trials

and hence the sum of n iid Bernoulli random variables.

D.R.V: Geometric Random Variable

#	Property	Description
1	S_X	$S_X = 0, 1, ..., n$
2	p_k	$p_k = p \cdot (1-p)^k,\ k = 0, 1, ..n$
3	$E[X]$	$E[X] = \frac{1-p}{p}$
4	Var[X]	Var $[X] = \frac{1-p}{p^2}$
5	$G(z)$	$G(z) = \frac{p}{1-qz}$

X is the number of failures before the first success in a sequence of iid Bernoulli trials. It's the only discrete r.v. with the memoryless property.

Second Version of the Geo. R.V.

#	Property	Description
6	NOTE!	$S_X = 1, ..., n$
7	p_k	$p_k = p(1-p)^{k-1},\ k = 1, ..n$
8	$E[X]$	$E[X] = 1/p$
9	Var[X]	Var $[X] = \frac{1-p}{p^2}$
10	$G(z)$	$G(z) = \frac{pz}{1-qz}$

D.R.V: Poisson Random Variable

#	Property	Description
1	S_X	$S_X = 0, 1, ...$
2	p_k	$p_k = \frac{\lambda^k}{k!} e^{-\lambda},\ k = 0, 1, ...$ and $\lambda > 0$
3	$E[X]$	$E[X] = \lambda$
4	Var[X]	Var $[X] = \lambda$
5	$G(z)$	$G(z) = e^{\lambda(z-1)}$

X is the number of events that occur in one time period

when the time between events is exp. distr. with mean $1/\lambda$

C.R.V: Uniform Random Variable

#	Property	Description
1	S_X	$S_X = [a, b]$
2	$f(x)$	$f(x) = \frac{1}{b-a},\ a \leq x \leq b$
3	$E[X]$	$E[X] = \frac{a+b}{2}$
4	Var[X]	Var $[X] = \frac{(b-a)^2}{12}$
5	$\Phi(\omega)$	$\Phi(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$

C.R.V: Exponential Random Variable

#	Property	Description
1	S_X	$S_X = [0, \infty)$
2	$f(x)$	$f(x) = \lambda e^{-\lambda x},\ x \geq 0,\ \lambda > 0$ ← "Lebesgue density"
3	$E[X]$	$E[X] = \frac{1}{\lambda}$
4	Var[X]	Var $[X] = \frac{1}{\lambda^2}$
5	$\Phi(\omega)$	$\Phi(\omega) = \frac{\lambda}{\lambda - j\omega}$

Baum & Breuer, p. 260

LST (Laplace-Steltjees Xform)

$\mathcal{L}[f(x)] = \frac{\lambda}{\lambda+s}$

..the only c.r.v. with the memoryless property.

C.R.V: Gaussian Random Variable

#	Property	Description
1	S_X	$S_X = (-\infty, +\infty)$
2	$f(x)$	$f(x) = \frac{exp(-(x-m)^2/(2\sigma^2))}{\sqrt{2\pi}\sigma},\ -\infty < x < +\infty,\ \sigma > 0$
3	$E[X]$	$E[X] = m$
4	Var[X]	Var $[X] = \sigma^2$
5	$\Phi(\omega)$	$\Phi(\omega) = exp(jm\omega - \sigma^2\omega^2/2)$

Approximates the sum of a large # of independent rvs.