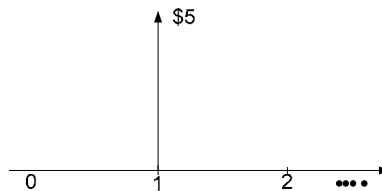


ISM 50
Fall 06
Note on Cash Flows and Rate of Return

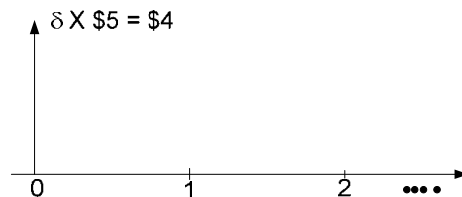
Time Value of Money

A cash flow is a series of payments or receipts spaced out in time. The key concept in analyzing cash flows is that receiving a \$1 today is more desirable than receiving a \$1 some time in the future. How much more desirable \$1 today is compared to \$1 in the future depends on the person or firm's view point. The person or firm, which we will refer to as the decision maker, quantifies what she thinks \$1 in the future is worth by deciding what quantity of money, received today, would be equally desirable.

For example, the decision maker may believe, "\$1 one year from now is as desirable as receiving \$0.80 today." To formalize her logic, the decision maker defines a *discount factor* δ and in our example sets it equal to 0.80. It's important to remember that this is her discount factor, and other people's discount factor might be different. (For example someone else might value \$1 one year from now as much as \$0.90 today and pick a discount factor of 0.90.) If someone asked her how much she values \$5 one year in the future, she would take \$5 times the discount factor $\delta=0.80$ which yields \$4. Often it is helpful to visualize a cash flow on a timeline. For example, a cash flow where \$5 is received one year from now would be drawn as:

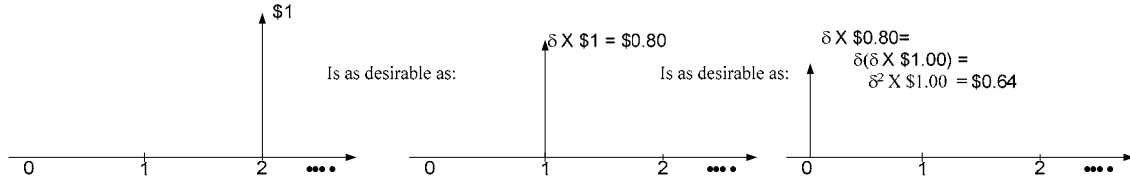


The horizontal axis is in units of years. It is common convention to label the current year as "0." Because the decision maker's discount factor δ is 0.80, the above cash flow is equally desirable as the cash flow:



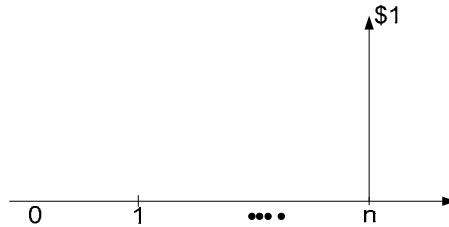
Net Present Value

The same approach can be used to compute how desirable it would be to receive \$1 two years from now.



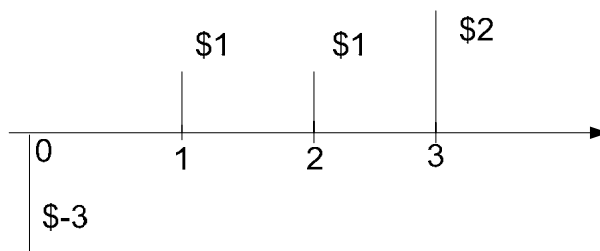
In the above figure, we multiply the \$1 in year 2 by δ to determine that it would be equally desirable to receive \$0.80 a year earlier, and in this case a year earlier would be year 1. Then we multiply \$0.80 by the discount factor delta to determine that is equally desirable to receive \$0.64, a year earlier, which would be year 0. This calculation is equivalent to taking original amount \$1, and multiplying it by δ^2 . Thus it is equally desirable to receive \$0.64 in two years as it is to receive \$1 today. What we have computed is what is called a *Net Present Value (NPV)*. The net present value of a cash flow is a quantity of money, which if received today, would be equally desirable as the cash flow. So the cash flow of receiving \$1 in year 2, has an NPV of $\delta^2 = \$0.64$. Note that the answer depends on the value of our discount factor δ .

We can generalize the above example, by computing the NPV of \$1 received in year n .



\$1 in year n is as desirable as δ in year $n-1$, or as desirable as δ^2 in year $n-2$, and so on. By this logic, \$1 in year n is as desirable as δ^n . Thus, the NPV of $\$x_n$ received in year n is $x_n\delta^n$. A cash flow may have payments or receipts in multiple years. Consider the cash flow:

Figure 1:



To analyze the NPV of such a cash flow, one approach is to compute the NPV of each payment or receipt, and then add them together. For example the NPV of the -\$3 payment

in year 0 has an NPV of \$-3. No discount factor needs to be applied because the payment happens in year 0, which is “the present.” The NPV of the \$1 receipt in year 1 has an NPV of δ . The NPV of the \$1 receipt in year 2 has an NPV δ^2 . Finally the NPV of the \$2 receipt in year 3 has an NPV of $2 \times \delta^3$. Thus the overall cash flow has an NPV of

$$-3 + \delta + \delta^2 + 2\delta^3.$$

When the discount factor is set to 0.80, the above expression works out to be -\$0.536. Imagine that the above cash flow were the cash flow of particular project. (i.e. You plan to spend \$3 million today, get \$1 million of benefit in year 1 and 2 each, and \$2 million in year 3.) The above analysis says that this project’s NPV is -\$536,000 – which suggests you should not proceed with the project. However, you should remember that this is all dependent on your discount factor δ , which is a reflection of how much you value money received in the future. Imagine your discount factor was 1, how would your analysis change?

Using the same reasoning as we used in the above example, we can derive a more general formula. A receipt of $\$x_0$ in year 0 has an NPV of x_0 . A receipt of $\$x_1$ in year 1 has an NPV of δx_1 . A receipt of $\$x_2$ in year 2 has an NPV of $\delta^2 x_2$. A receipt of $\$x_n$ in year n has an NPV of $\delta^n x_n$. Thus if we have a cash flow for which we receive of $\$x_0$ in year 0, $\$x_1$ in year 1, and $\$x_2$ in year 2, and so on, the NPV of the entire cash flow is:

$$\text{NPV} = x_0 + \delta x_1 + \delta^2 x_2 + \delta^3 x_3 + \dots = \sum_{j=0}^{\infty} \delta^j x_j \quad (1)$$

Relation of Interest Rate and Discount Factor

Oftentimes, a decision maker’s discount factor is based on the interest rate or rate of return she could receive if she were to invest her capital in the bank or perhaps in another project. For example, suppose the *interest rate* i , were 0.10 (10%). Then having \$1 in the bank today would be worth $\$1 \times (1+i) = \1.10 a year from now. Conversely, receiving \$1.10 a year from now would be as desirable as having \$1 today that could be put in the bank. Thus the NPV of getting $\$x$ one year from now is $\$x / (1+i)$. But we have also said that the NPV of getting $\$x$ one year from now is δx . Setting these two expressions to be equal to each other, we can derive the relationship between the discount factor and the interest rate

$$\delta = \frac{1}{1+i}. \quad (2)$$

If we substitute this relation between interest rate i and discount factor δ into equation (1), we get an expression for NPV in terms of the interest rate i .

$$\text{NPV} = x_0 + (1+i)^{-1} x_1 + (1+i)^{-2} x_2 + (1+i)^{-3} x_3 + \dots = \sum_{j=0}^{\infty} (1+i)^{-j} x_j \quad (3)$$

In summary, if you are given an interest rate i and asked to compute NPV, you should use equation (3). If you are given a discount factor δ , and asked to compute NPV, you should use equation (1). If you do not have either a discount factor or an interest rate, you do not have enough information to compute the NPV.

Discount Rate

In some problems, you may be given what is called a *discount rate*. The discount rate plays the same role as the interest rate in the above discussion, in that it describes by what percentage we should discount future payments. For example, if you are told that the discount rate i is 0.25 (25%), the corresponding discount factor δ is

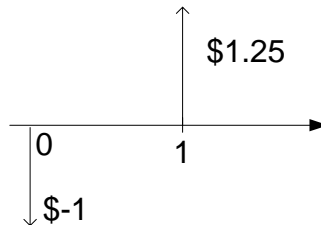
$$\delta = \frac{1}{1+i} = \frac{1}{1+0.25} = 0.8. \quad (2)$$

Rate of Return or Return on Investment

The return on investment, or more commonly called the rate of return (ROR), is an inverse problem to computing the NPV. When one computes the ROR on an investment decision one is essentially asking, “What would the interest rate at the bank¹ have to be in order for me to be neutral about investing in my project?” Thus, we are trying to find an interest rate i such that the NPV of the project is 0, as expressed by the equation:

$$x_0 + (1+i)^{-1}x_1 + (1+i)^{-2}x_2 + (1+i)^{-3}x_3 + \dots = \sum_{j=0}^{\infty} (1+i)^{-j}x_j = 0 \quad (4)$$

For example consider the cash flow



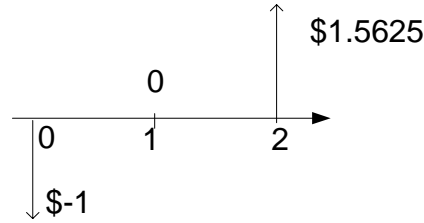
One’s intuition might immediately suggest that the ROR is 25%, but let us verify this. We want to compute an interest rate i for which we are neutral about the investment, meaning that we find an i for which the NPV is 0. We use equation (4), to get

$$-1 + (1+i)^{-1}1.25 = 0.$$

We can easily solve this to find $i = 0.25$. Thus the ROR is 25%, which means if the interest rate at the bank were 25%, we would be neutral about proceeding with a project that required \$1 investment today and would yield \$1.25 in revenue tomorrow. This is because we could also invest that \$1 in the bank and get exactly the same return.

Now consider the cash flow

¹ More generally, the question one asks is “What would the rate of return have to be for other investment alternatives (and one such alternative is putting the money in the bank) for me to neutral about investing in the project?”



Again we use equation (4) to get

$$-1 + (1+i)^{-2}1.5625 = 0$$

$$(1+i)^2 = 1.5625$$

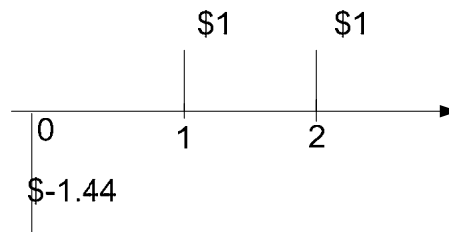
$$(1+i) = 1.25$$

$$i = 0.25$$

Thus the ROR of this cash flow is also 0.25. Again suppose the cash flow represents what would happen if we invested in a project. The intuition is that if we put \$1 in the bank that paid 25% interest, instead of doing the project, in 2 years that \$1 would be worth $(1.25)^2 = \$1.5625$, which is the same outcome we would get if we had done the project. Thus when the interest rate is 0.25, we are neutral between the alternatives of doing the project or investing in the bank.

Now consider the cash flow depicted below

FIGURE 2:



Again, suppose this cash flow is what would happen if we decided to invest in a project. We want to find the ROR, which is the interest rate for which the NPV is 0. Using equation (4), we can write the equation

$$-1.44 + (1+i)^{-1}1 + (1+i)^{-2}1 = 0$$

$$(1+i)^{-2}1 + (1+i)^{-1}1 - 1.44 = 0$$

$$(1+i)^{-1} = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times -1.44}}{2}$$

$$(1+i)^{-1} = \frac{-1 \pm 2.6}{2}$$

$$(1+i)^{-1} = 0.8 \text{ or } -1.8$$

$$i = 0.25 \text{ or } -1.5555$$

By dropping the nonsensical answer of -1.5555, we find that the ROR is 0.25. Thus when the interest rate available at the bank, or in other investment opportunities, is 0.25 we are neutral about proceeding with a project that has the cash flow of Figure 2. To further see

why, suppose we put \$1.44 in the bank in year 0 – instead of investing in the project with the cash flow shown in Figure 2. With an interest rate of 0.25, the \$1.44 would become $\$1.44 \times 1.25 = \1.80 by year 1. In year 1, we could withdraw \$1 from the bank, and keep the remaining \$0.80 in the account. In year 2, that \$0.80 would grow to $\$0.80 \times 1.25 = \1.00 , and then we could withdraw that dollar. By doing this, we were able to duplicate the cash flow in Figure 2 by investing in the bank instead of investing in the project.

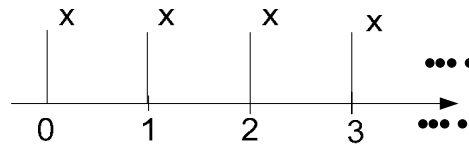
Now suppose the interest rate at a bank were 0.30. Because the ROR of the project whose cash flow is illustrated by Figure 2 has only 0.30, we should prefer to invest in the bank rather than the project. This can also be verified by NPV analysis. If we compute the NPV using equation (3) we find that the NPV is

$$-1.44 + (1 + 0.3)^{-1}1 + (1 + 0.3)^{-2}1 = -0.0791$$

which is negative. This confirms that we should not invest in the project when the interest rate is 0.30.

Infinite Series

Sometimes a series of payments or receipts is never ending. One special case is when the same payment or receipt is made or received every year forever.



The NPV of this infinite series of receipts can be put into a closed form expression by the using equation (1) and making the following manipulations:

$$x + \delta x + \delta^2 x + \delta^3 x + \dots = NPV$$

$$\delta x + \delta^2 x + \delta^3 x + \delta^4 x \dots = \delta \times NPV$$

$$x = (1 - \delta)NPV$$

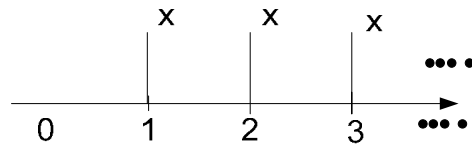
$$NPV = \frac{1}{1 - \delta} x$$

(Here we have taken the first equation, multiplied both sides by δ to get the second equation, and then subtracted the second equation from the first equation to get the third equation.)

Furthermore, the NPV can also be expressed in terms of an interest rate i instead of the discount factor δ by substituting equation (2).

$$\begin{aligned} NPV &= \frac{1}{1 - (1 + i)^{-1}} x \\ &= \frac{1 + i}{i} x = \left(1 + \frac{1}{i}\right) x \end{aligned}$$

The above analysis assumed that the cash flow begins in year 0. If we assume the cash flow begins in year 1, then we can simply subtract the NPV of the year 0 cash flow, which is just x , from the above expression to find that for the cash flow



the NPV is

$$NPV = \left(1 + \frac{1}{i}\right)x - x = \frac{1}{i}x$$