Definition. Let G be a CFG, then $L(G)$ is the language generated by G. If L is a language than L' is its complement.

8.1) Use the Pumping Lemma to show that the following languages are not a context-free.
a) $L = \{a^i b^j c^k \mid i < j < k\}$
Hint: let $u = a^n b^{n+1} c^{n+2}$. If $wy = c^l$ let $m = 0$, otherwise let $m \geq 2$ ($vwxyz = u$).
c) $L = \{a^n b^{2n} c^n \mid n \geq 0\}$
Hint: let $u = a^n b^{2n} c^n$ and $m \neq 1$.
e) $L = \{a^n b^m a^n b^{n+m} \mid m, n \geq 0\}$
Hint: let $u = a^n b^n a^n b^{2n}$ and $m \neq 1$.

8.5)
a) Claim: $L = \{a^n b^m a^m b^n \mid m, n \geq 0\}$ is context-free.
Proof: If G is the CFG with the following productions: $S \to aSb \mid T \; T \to bTa \mid \Lambda$, then $L(G) = L$.
b) Claim: $L = \{xayb \mid x, y \in \{a, b\}^*, \; |x| = |y|\}$ is context-free.
Proof: If G is the CFG with the following productions: $S \to Tb \mid Tb \, T \to aTa \mid aTb \mid bTa \mid bTb \mid a$, then $L(G) = L$.
c) Claim: $L = \{xcx \mid x \in \{a, b\}^*\}$ is not context-free.
Hint: use the Pumping Lemma with $u = a^n b^n c a^n b^n$and $m \neq 1$.
d) Claim: $L = \{xyx \mid x, y \in \{a, b\}^* |x| \geq 1\}$ is context-free.
Proof: If G is the CFG with the following productions: $S \to aTa \mid bTb \; T \to aT \mid bT \mid \Lambda$, then $L(G) = L$.

8.8)
a) $L = \{a^i b^j c^k \mid i \geq j \text{ or } i \geq k\}$ is context-free, but it's complement is not.
Proof: If G is the CFG with the following productions: $S \to aS \mid aSc \mid C \mid B \; B \to Bb \mid \Lambda$, $C \to Cc \mid D$, $D \to aDb \mid \Lambda$, then $L(G) = L$.
To show that $L' = \{a, b, c\}^* \setminus L$, the complement of L, is not context-free use Ogden's Lemma with the string $u = a^n b^{n+1} c^{n+1}$ with only the last $n+1$ positions distinguished. Then wy doesn't contain any a's so that setting $m = 0$ transforms u into a string in L.

8.10)
a) Claim: Let L be a CFL and F be finite. Then $L - F$ is a CFL.
Proof: F is finite, and hence regular, so that F' is regular. Hence, $L - F = L \cap F'$ is a CFL by Theorem 8.4.
b) Claim: Let L be a non CFL and F be finite. Then $L - F$ is not a CFL.
Proof: Towards a contradiction assume that L is not a CFL and F is finite, but that $L - F$ is a CFL. F is finite, and hence regular (and context-free) so $L = (L - F) \cup F$ is a CFL, since CFL's are closed under union. But this contradicts the assumption that L is not a CFL.
c) Claim: Let L be a non CFL and F be finite. Then $L \cup F$ is not a CFL.
Proof: Towards a contradiction assume that L is not a CFL and F is finite, but that $L \cup F$ is a CFL. F is finite, and hence regular, so that F' is regular. Hence, $L - F = (L \cup F) \cap F'$ is a CFL by Theorem 8.4. But this contradicts (b).