Note: Whenever binary set operators are used on regular expressions they will be defined for the purposes of these problems as acting on the corresponding regular languages, e.g. for regular expressions t and u, $t \subseteq u$ whenever the regular language corresponding to t is a subset of the regular language corresponding to u.

2.59) Define the function $rev(x) = x^r$ such that
1) $\Lambda^r = \Lambda$, and
2) $(xa)^r = ax^r$ for any $x \in \Sigma^*$and any $a \in \Sigma$.
Prove the following facts about $rev : \Sigma^* \to \Sigma^*$.
a) $[(xy)^r = y^r x^r$ for any strings $x, y \in \Sigma^*]$

Proof. We proceed by induction on the length of y. The base case, $|y| = 0$: $(xy)^r = (x\Lambda)^r = x^r = \Lambda^r x^r = y^r x^r$. Hence, it suffices to show that for any $n \geq 0$, if $(xy)^r = y^r x^r$ for any strings $x, y \in \Sigma^*$ such that $|y| = n$, then $(wz)^r = z^r w^r$ for any strings $w, z \in \Sigma^*$ such that $|z| = n + 1$. Let $z \in \Sigma^*$such that $|z| = n + 1$. Without loss of generality we may assume that $z = ay$ where $a \in \Sigma$, $y \in \Sigma^*$, and $|y| = n$. Then $(wz)^r = (way)^r = y^r(wa)^r = y^r a w^r = (ay)^r w^r = z^r w^r$. □

b) $[(x^r)^r = x$ for any string $x \in \Sigma^*]$

Proof. By induction on the length of x. The base case $|x| = 0$ is covered by (1) in the definition of rev. Hence, it suffices to show that for any $n \geq 0$, if $(x^r)^r = x$ for any string $x \in \Sigma^*$ such that $|x| = n$, then $(z)^r = z$ for any string $z \in \Sigma^*$ such that $|z| = n + 1$. Without loss of generality we may assume that $z = ax$ where $a \in \Sigma$, $x \in \Sigma^*$, and $|x| = n$. Then, using the results from part (a), $(z^r)^r = ((ax)^r)^r = (x^r a)^r = a(x^r)^r = ax = z$. □

c) $[(x^n)^r = (x^r)^n$ for any string $x \in \Sigma^*$ and any $n \geq 0]$

Proof. By induction on n. The base case, $n = 0$: $(x^n)^r = (x^0)^r = (\Lambda)^r = \Lambda = (x^r)^0 = (x^r)^n$. Hence, it suffices to show that $(x^n)^r = (x^r)^n$ for any string $x \in \Sigma*$ assuming that $(x^{n-1})^r = (x^r)^{n-1}$. Then, using the result from part (a), $(xx^{n-1})^r = (x^{n-1})^r x^r = (x^r)^{n-1} x^r = (x^r)^n$. □

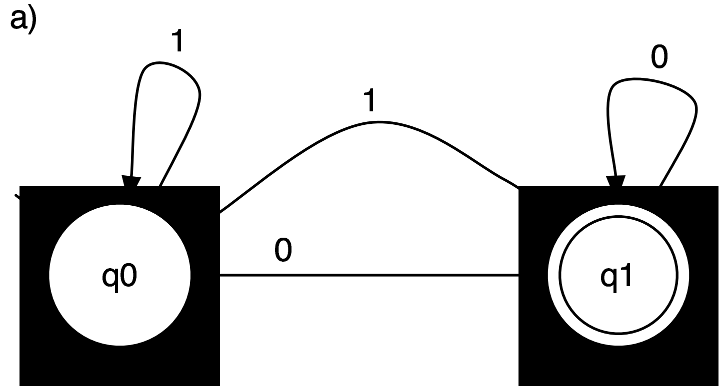
3.4) Show that $(111^*)^* = (11 + 111)^*$.

Proof. Assume that $\Sigma = \{1\}$. Since $\Lambda \in (111^*)^* \cap (11 + 111)^*$, it suffices to show that $1^n \in (11 + 111)^*$ and $1^n \in (111^*)^*$ whenever $n \geq 2$.
[$1^n \in (111^*)^*$ whenever $n \geq 2$]: $1^n \in 111^* \subseteq (111^*)^*$ whenever $n \geq 2$.
[$1^n \in (11 + 111)^*$ whenever $n \geq 2$]: If n is even, then $1^n \in (11)^* \subseteq (11 + 111)^*$. If $n \geq 2$ is odd, then $n - 3 \geq 0$ is even and $1^3 = 111 \in (11 + 111)^*)$, hence $1^{n-3}, 1^3 \in (11 + 111)^*$. But Kleene star is closed under concatenation, i.e. $t^* \supseteq uv$ whenever $t^* \supseteq u$ and $t^* \supseteq$v, so that $1^n = 1^{n-3}1^3 \in (11 + 111)^*$. □

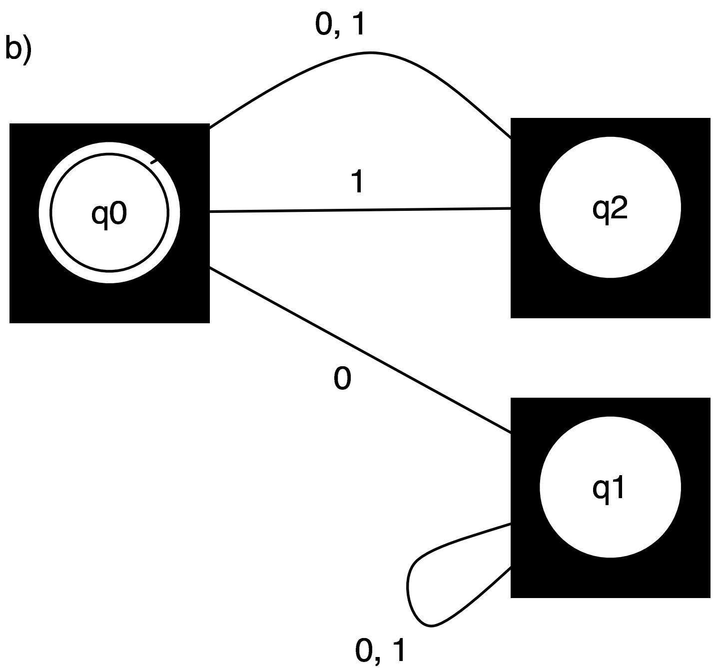
3.7) Describe the smallest set of languages that contain the basic languages, $\emptyset, \{\Lambda\}$, and $\{a\}$ (for every $a \in \Sigma$), and is closed under the following operations.
a) union: The set of all finite languages consisting of elements of Σ, i.e. strings consisting of a single element Σ.
b) concatenation: The set of all languages containing a single finite length string composed of elements of Σ.
d) union and concatenation: The set of all finite languages.

3.20)

a)

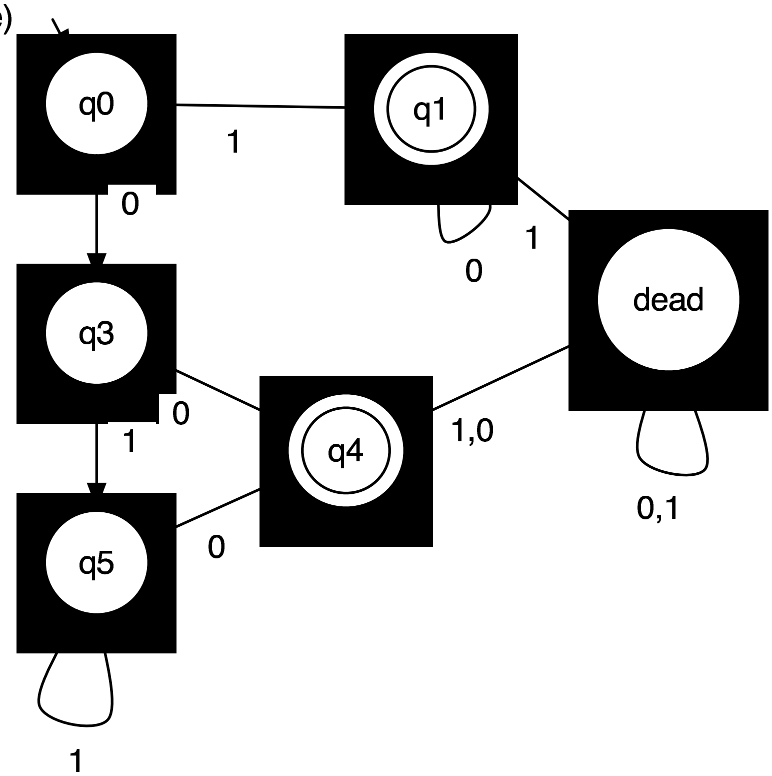


b)



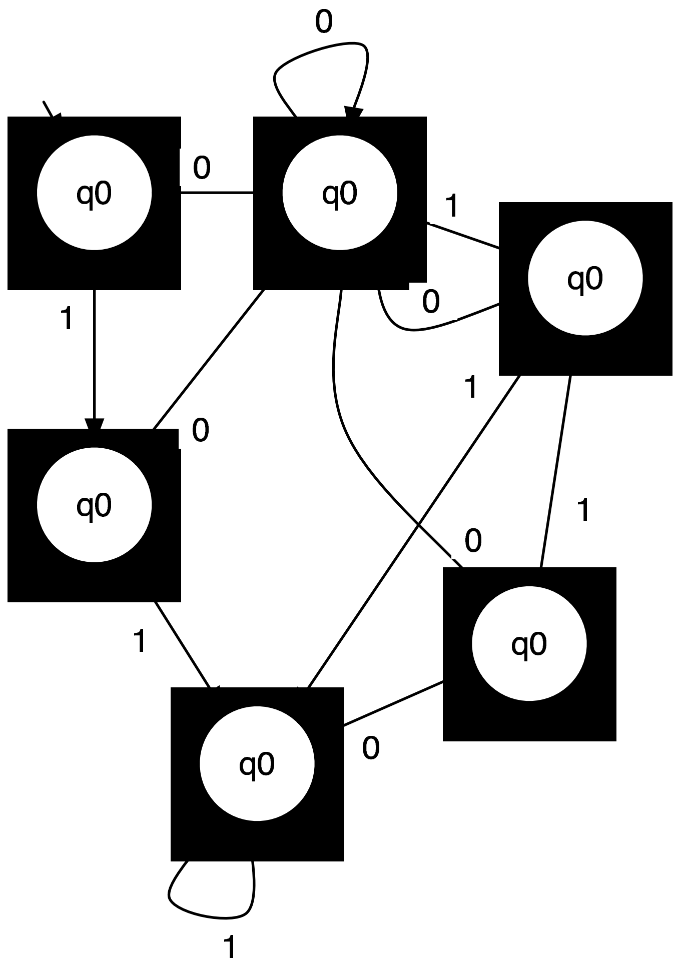
3.20)

e)



3.20)

f)



3.23) If two FA's are said to be equivalent if they accept the same language, then for every FA, there exists another equivalent FA with more states. Explain why this is true.

We can always add an unreachable state. More precisely, let $\{Q, \Sigma, q_0, A, \delta\}$ be an arbitrary FA, let $q \notin Q$ be a *new* state, and define $\delta(q, a) = q$ for every $a \in \Sigma$. Then $\{Q \cup \{q\}, \Sigma, q_0, A, \delta\}$ is equivalent to $\{Q, \Sigma, q_0, A, \delta\}$, but with one more state.

3.34) Let r and s be regular expressions over Σ. Prove that $(r^*s^*)^* = (r+s)^*$.

Proof. It suffices to show that $(r^*s^*)^* \subseteq (r+s)^*$ and $(r^*s^*)^* \supseteq (r+s)^*$.

$[(r^*s^*)^* \supseteq (r+s)^*]$: $r^*s^* \supseteq r^*\Lambda \supseteq r$ and $r^*s^* \supseteq \Lambda s^* \supseteq s$ which implies that $r^*s^* \supseteq r+s$. But, by the definition of the Kleene star $t^* \supseteq u^*$ whenever $t \supseteq u$, so that $(r^*s^*)^* \supseteq (r+s)^*$.

$[(r^*s^*)^* \subseteq (r+s)^*]$: We proceed by induction.

Base cases: $(r+s)^* \supseteq \Lambda = (r^*s^*)^0$. Additionally $(r+s)^* \supseteq r^*$ and $(r+s)^* \supseteq s^*$. But Kleene star is closed under concatenation, i.e. $t^* \supseteq uv$ whenever $t^* \supseteq u$ and $t^* \supseteq$v, so that $(r+s)^* \supseteq r^*s^* = (r^*s^*)^1$.

Hence, it suffices to show that for any $i \geq 1$, $(r^*s^*)^* \subseteq (r+s)^{i+1}$ whenever $(r^*s^*)^* \subseteq (r+s)^i$. But , since Kleene star is closed under concatenation, $(r^*s^*)^* \subseteq (r+s)^i(r+s)^1 = (r+s)^{i+1}$. □