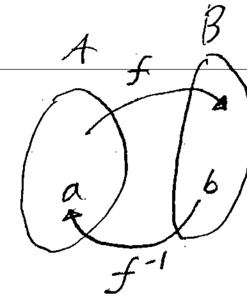
CMPS 130, Fall 08, HW1 — Krishna Roskin

1.27

15 pts

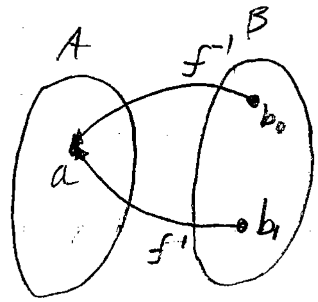
To show that f^{-1} is a bijection we must show that f^{-1} is onto and one-to-one.



To show onto:

Let $a \in A$ and $b = f(a)$. By def. of f^{-1} $f^{-1}(b) = a$. Thus every point in A is "hit" by f^{-1}!

To show one-to-one:



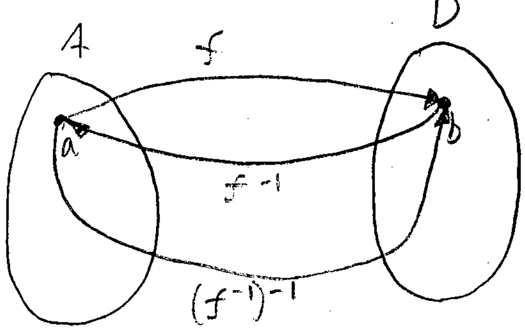
Let $b_0, b_1 \in B$ and $f^{-1}(b_0) = f^{-1}(b_1)$. Let $a = f^{-1}(b_0) = f^{-1}(b_1)$. By def. $f(a) = b_0$ and $f(a) = b_1$ and thus $b_0 = b_1$ and so f^{-1} is one to one.

Thus f^{-1} is a bijection.

Now consider $(f^{-1})^{-1}$, which is well defined because we proved that f^{-1} is a bijection.

Let $a \in A$ and $b = f(a)$. Thus $f^{-1}(b) = a$. By def. off an inverse, $(f^{-1})^{-1}(a) = b$.

Thus $(f^{-1})^{-1}$ sends ever point a to the same point b that f sends. Thu $(f^{-1})^{-1} = f$.



$\boxed{1.35}$

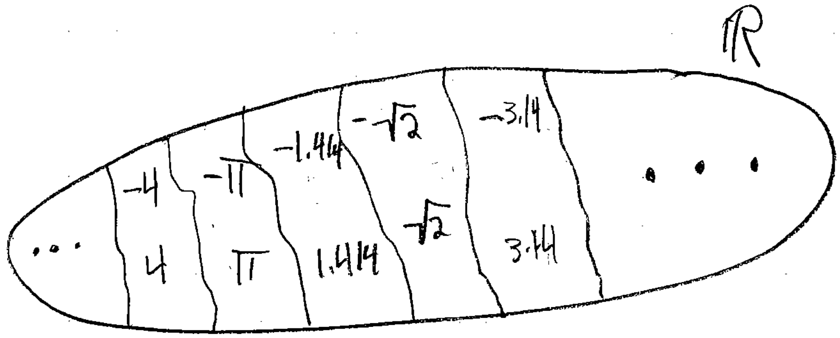
10 pts

Let $x \in \mathbb{R}$. Then

$$[x]_R = \{y \in \mathbb{R} \mid x R y\}$$
$$= \{y \in \mathbb{R} \mid f(x) = f(y)\}$$
$$= \{y \in \mathbb{R} \mid x^2 = y^2\}$$

Because $y = \pm x$ are the only solutions to $x^2 = y^2$, we have that

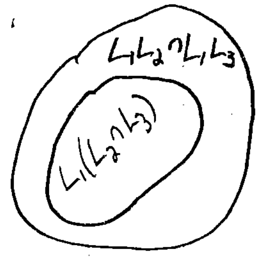
$$[x]_R = \{x, -x\}.$$



$\boxed{1.47}$

(a) 10pts

Let $w \in L_1(L_2 \cap L_3)$. Thus w has the form $w_1 w_2$ where $w_1 \in L_1$ and $w_2 \in L_1$ and $w_2 \in L_2$. Thus $w = w_1 w_2 \in L_1 L_2$ and $w \in L_1 L_3$. Therefore, $w \in L_1L_2 \cap L_1L_3$. So $L_1(L_2 \cap L_3) \subseteq L_1L_2 \cap L_1L_3$.



To see that the subset <u>can</u> be strict, consider:

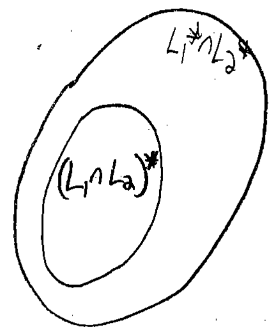
$$L_1 = \{0, 01\}, \quad L_2 = \{1\}, \quad L_3 = \{11\} \Rightarrow L_2 \cap L_3 = \emptyset, \quad L_1L_2 = \{01, 011\}, \quad L_1L_3 = \{011, 0111\}$$

Thus $L_1(L_2 \cap L_3) = \emptyset$ and $L_1L_2 \cap L_1L_3 = \{011\}$. So, in this case $L_1(L_2 \cap L_3) \subsetneq L_1L_2 \cap L_1L_3$.

$A \subsetneq B$ means $A \subseteq B$ but $A \neq B$.

(b) 10pts

Let $w \in (L_1 \cap L_2)^*$, then $w = w_1 w_2 \cdots w_n$ for $n \geq 0$, where $w_i \in L_1 \cap L_2$ where $i = 0, 1, \ldots, n$. Since w_i is in the intersection, $w_i \in L_1$ <u>and</u> $w_i \in L_2$. Thus $w \in L_1^*$ and $w \in L_2^*$ and so $w \in L_1^* \cap L_2^*$.



To see that the inclusion can be strict consider

$$L_1 = \{aa\}, \quad L_2 = \{aaa\} \Rightarrow L_1^* = \{a^{2n} \mid n \in \mathbb{N}\}, \quad L_2^* = \{a^{3n} \mid n \in \mathbb{N}\}, \quad L_1 \cap L_2 = \emptyset$$

Thus $(L_1 \cap L_2)^* = \emptyset^* = \{\Lambda\}$ but $L_1^* \cap L_2^* = \{a^{6n} \mid n \in \mathbb{N}\}$. For example $aaaaaa = a^6 \in L_1^* \cap L_2^*$ but $a^6 \notin (L_1 \cap L_2)^*$.

$\boxed{1.47}$ con't

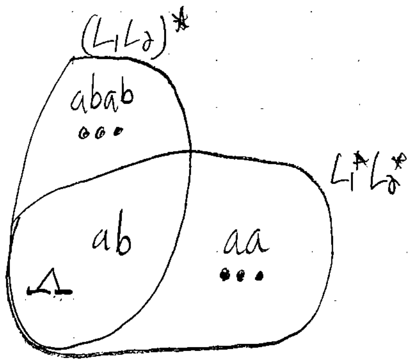
(c) Consider

10 pts

$L_1 = \{a\} \Rightarrow L_1^* = \{a^n \mid n \in \mathbb{N}\}$
$L_2 = \{b\} \Rightarrow L_2^* = \{b^m \mid m \in \mathbb{N}\}$

Then $L_1L_2 = \{ab\}$ and $(L_1L_2)^* = \{(ab)^n \mid n \in \mathbb{N}\}$
and $L_1^* L_2^* = \{a^n b^m \mid n \in \mathbb{N}, m \in \mathbb{N}\}$.

The string abab is in $(L_1L_2)^*$ but not in $L_1^* L_2^*$. On the other hand, aa is in $L_1^* L_2^*$ but not $(L_1L_2)^*$.



Thus there is no general subset relation.

1.33e

15pts

To show that R is an equivalence we must show that it is reflexive, symmetric, and transitive.

Reflexive: Let $x = x_0 x_1 \cdots$. $x_j = x_j$ for all j and in perticular at $j \geq k = 0$. Thus $x R x$.

Symmetric: Suppose, $x R y$. Then $\exists k$ such that

$$\begin{aligned} x &= x_0 x_1 \cdots x_k x_{k+1} \cdots \\ y &= y_0 y_1 \cdots y_k y_{k+1} \cdots \end{aligned}$$

equal after k

Thus $y_j = x_j$ for $j \geq k$ and so $y R x$.

Transitive: Suppose $x R y$ and $y R z$. Since $x R y$, $\exists k_0$ such that $x_j = y_j$ for $j \geq k_0$. Since $y R z$, $\exists k_1$ such that $y_j = z_j$ for $j \geq k_1$.

$$\begin{aligned} x &= x_0 x_1 x_2 \cdots x_{k_0-1} x_{k_0} x_{k_0+1} \cdots x_{k_1-1} x_{k_1} x_{k_1+1} \cdots \\ y &= x_0 x_1 x_3 \cdots y_{k_0-1} y_{k_0} y_{k_0+1} \cdots y_{k_1-1} y_{k_1} y_{k_1+1} \cdots \\ z &= z_0 z_1 z_2 \cdots z_{k_0-1} z_{k_0} z_{k_0+1} \cdots z_{k_1-1} z_{k_1} z_{k_1+1} \cdots \end{aligned}$$

x_i equal y_i

y_i equal z_i

x_i, y_i, z_i all equal

(this only shows $k_0 \leq k_1$ case)

Let $k = \max\{k_0, k_1\}$. Then $x_j = y_j$ for $j \geq k$ since $k \geq k_0$ and $y_j = z_j$ for $j \geq k$ since $k \geq k_1$. Thus $x_j = z_j$ for $j \geq k$. Therefore $x R z$.

Thus R is an equivalence relation.

$\boxed{2.22}$

15 pts

We are asked to show that

$$L^+ = \bigcup_{i=1}^{\infty} L^i$$

is a subset of L. To show this, we will show that $L^i \subseteq L$ for $i \geq 1$. Thus the union will be a subset of L.

<u>Lemma</u> $L^i \subseteq L$ for $i \geq 1$.

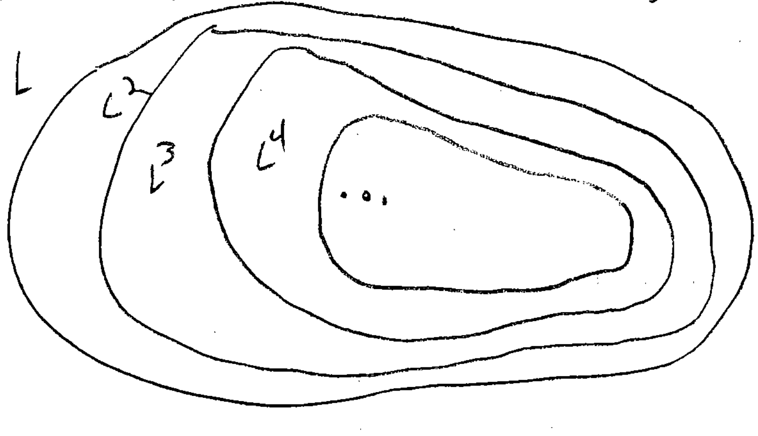
<u>Proof</u> The proof will be by induction on i. The base cases are $i=1$ and $i=2$. For $i=1$, $L \subseteq L$ trivially. For $i=2$, $L^2 \subseteq L$ by the hypothesis on L.

Now assume that $L^k \subseteq L$ for some $k > 2$. Consider

$$L^{k+1} = L^k L$$

Suppose $w \in L^{k+1}$. Thus w can be written as $w = xy$ were $x \in L^k$ and $y \in L$. By the induction hypothesis, $L^k \subseteq L$ and thus $x \in L$. Thus $w = xy \in L^2 \subseteq L$. Thus $L^{k+1} \subseteq L$. This completes the induction proof. ∎

Given the assumption that $L^2 \subseteq L$, we have shown that $L^i \subseteq L$ for $i \geq 1$, thus thus the union is also a subset of L, i.e. $L^+ \subseteq L$.



2.53a

15pts

Note that for this problem we use this defination of the Fibonacci function

$f_0 = 0$
$f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

The proof will be by strong induction n. Since there are two base cases in the recursive defination of f_n, there are two base cases in the induction proof:

$$n=0: \; c(a^0 - b^0) = c(1-1) = 0 = f_0$$

$$n=1: \; c(a^1 - b^1) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$
$$= \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right) = 1 = f_1$$

Now assume that $f_i = c(a^i - b^i)$ for $0 \leq i \leq k$ for some $k \geq 2$. Consider

$$f_{k+1} = f_k + f_{k-1} \quad \text{(since } k \geq 2\text{)}$$
$$= c(a^k - b^k) + c(a^{k-1} - b^{k-1}) \quad \text{(by induction hypothesis)}$$
$$= c[a^{k-1}(a+1) - b^{k-1}(b+1)].$$

Now consider a^2 and b^2

$$a^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 \qquad b^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2$$
$$= \frac{1+2\sqrt{5}+5}{4} \qquad = \frac{1-2\sqrt{5}+5}{2}$$
$$= \frac{6+2\sqrt{5}}{4} \qquad = \frac{6-2\sqrt{5}}{4}$$
$$= 1+a \qquad = 1+b$$

Thus $f_{k+1} = c[a^{k-1}a^2 - b^{k-1}b^2] = c(a^{k+1} - b^{k+1})$
Which proves the induction step and proof.