

UNIVERSITY OF CALIFORNIA, SANTA CRUZ  
BOARD OF STUDIES IN COMPUTER ENGINEERING



CMPE 240: INTRODUCTION TO LINEAR DYNAMICAL SYSTEMS

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PROBLEM SET 6

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1. *Harmonic oscillator.* The system  $\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x$  is called a *harmonic oscillator*.
  - (a) Find the eigenvalues, resolvent, and state transition matrix for the harmonic oscillator. Express  $x(t)$  in terms of  $x(0)$ .
  - (b) Sketch the vector field of the harmonic oscillator.
  - (c) The state trajectories describe circular orbits, *i.e.*,  $\|x(t)\|$  is constant. Verify this fact using the solution from part (a).
  - (d) You may remember that circular motion (in a plane) is characterized by the velocity vector being orthogonal to the position vector. Verify that this holds for any trajectory of the harmonic oscillator. Use only the differential equation; do not use the explicit solution you found in part (a).
2. *Properties of the matrix exponential.*
  - (a) Show that  $e^{A+B} = e^A e^B$  if  $A$  and  $B$  commute, *i.e.*,  $AB = BA$ . The converse is also true, *i.e.*, if  $e^{A+B} = e^A e^B$  then  $A$  and  $B$  commute. (But it is hard to show.)
  - (b) Carefully show that  $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$ .
3. *Determinant of matrix exponential.*
  - (a) Suppose the eigenvalues of  $A \in \mathbf{R}^{n \times n}$  are  $\lambda_1, \dots, \lambda_n$ . Show that the eigenvalues of  $e^A$  are  $e^{\lambda_1}, \dots, e^{\lambda_n}$ . You can assume that  $A$  is diagonalizable, although it is true in the general case.
  - (b) Show that  $\det e^A = e^{\text{Tr } A}$ .

*Hint:*  $\det X$  is the product of the eigenvalues of  $X$ , and  $\text{Tr } Y$  is the sum of the eigenvalues of  $Y$ .
4. *Linear system with a quadrant detector.* In this problem we consider the specific system

$$\dot{x} = Ax = \begin{bmatrix} 0.5 & 1.4 \\ -0.7 & 0.5 \end{bmatrix} x.$$

We have a detector or sensor that gives us the sign of each component of the state  $x = [x_1 \ x_2]^T$  each second:

$$y_1(t) = \text{sgn}(x_1(t)), \quad y_2(t) = \text{sgn}(x_2(t)), \quad t = 0, 1, 2, \dots$$

where the function  $\text{sgn} : \mathbf{R} \rightarrow \mathbf{R}$  is defined by

$$\text{sgn}(a) = \begin{cases} 1 & a > 0 \\ 0 & a = 0 \\ -1 & a < 0 \end{cases}$$

There are several ways to think of these sensor measurements. You can think of  $y(t) = [y_1(t) \ y_2(t)]^T$  as determining which quadrant the state is in at time  $t$  (thus the name *quadrant detector*). Or, you can think of  $y(t)$  as a one-bit quantized measurement of the state at time  $t$ .

Finally, the problem. You observe the sensor measurements

$$y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad y(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Based on these measurements, what values could  $y(2)$  possibly take on?

In terms of the quadrants, the problem can be stated as follows.  $x(0)$  is in quadrant IV, and  $x(1)$  is also in quadrant IV. The question is: which quadrant(s) can  $x(2)$  possibly be in?

You do not know the initial state  $x(0)$ .

Of course, you must completely justify and explain your answer.

5. *Some basic properties of eigenvalues.* Show that

- (a) the eigenvalues of  $A$  and  $A^T$  are the same
- (b)  $A$  is invertible if and only if  $A$  does not have a zero eigenvalue
- (c) if the eigenvalues of  $A$  are  $\lambda_1, \dots, \lambda_n$  and  $A$  is invertible, then the eigenvalues of  $A^{-1}$  are  $1/\lambda_1, \dots, 1/\lambda_n$ ,
- (d) the eigenvalues of  $A$  and  $T^{-1}AT$  are the same.

*Hint:* you'll need to use the facts  $\det AB = \det A \det B$  and  $\det A^{-1} = 1/\det A$  (provided  $A$  is invertible).

6. *Characteristic polynomial.* Consider the characteristic polynomial  $\mathcal{X}(s) = \det(sI - A)$  of the matrix  $A \in \mathbf{R}^{n \times n}$ .

- (a) Show that  $\mathcal{X}$  is *monic*, which means that its leading coefficient is one:  $\mathcal{X}(s) = s^n + \dots$ .
- (b) Show that the  $s^{n-1}$  coefficient of  $\mathcal{X}$  is given by  $-\text{Tr } A$ . ( $\text{Tr } X$  is the *trace* of a matrix:  $\text{Tr } X = \sum_{i=1}^n X_{ii}$ .)

- (c) Show that the constant coefficient of  $\mathcal{X}$  is given by  $\det(-A)$ .
- (d) Let  $\lambda_1, \dots, \lambda_n$  denote the eigenvalues of  $A$ , so that

$$\mathcal{X}(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n).$$

By equating coefficients show that  $a_{n-1} = -\sum_{i=1}^n \lambda_i$  and  $a_0 = \prod_{i=1}^n (-\lambda_i)$ .

7. *Left eigenvector properties.* Suppose  $w$  is a left eigenvector of  $A \in \mathbf{R}^{n \times n}$  with real negative eigenvalue  $\lambda$ .

- (a) Find a simple expression for  $w^T e^{At}$ .
- (b) Let  $\alpha < \beta$ . The set  $\{z \mid \alpha \leq w^T z \leq \beta\}$  is referred to as a *slab*. Briefly explain this terminology. Draw a picture in  $\mathbf{R}^2$ .
- (c) Show that the slab  $\{z \mid 0 \leq w^T z \leq \beta\}$  is invariant for  $\dot{x} = Ax$ .

8. *Some Matlab exercises.* Consider the continuous-time system  $\dot{x} = Ax$  with

$$A = \begin{bmatrix} -0.1005 & 1.0939 & 2.0428 & 4.4599 \\ -1.0880 & -0.1444 & 5.9859 & -3.0481 \\ -2.0510 & -5.9709 & -0.1387 & 1.9229 \\ -4.4575 & 3.0753 & -1.8847 & -0.1164 \end{bmatrix}.$$

- (a) What are the eigenvalues of  $A$ ? Is the system stable? You can use the command `eig` in Matlab.
- (b) Plot a few trajectories of  $x(t)$ , *i.e.*,  $x_1(t)$ ,  $x_2(t)$ ,  $x_3(t)$  and  $x_4(t)$ , for a few initial conditions. To do this you can use the matrix exponential command in Matlab `expm` (*not* `exp` which gives the element-by-element exponential of a matrix), or more directly, the Matlab command `initial` (use `help initial` for details.) Verify that the qualitative behavior of the system is consistent with the eigenvalues you found in part (8a).
- (c) Find the matrix  $Z$  such that  $Zx(t)$  gives  $x(t + 15)$ . Thus,  $Z$  is the ‘15 seconds forward predictor matrix’.
- (d) Find the matrix  $Y$  such that  $Yx(t)$  gives  $x(t - 20)$ . Thus  $Y$  reconstructs what the state was 20 seconds ago.
- (e) Briefly comment on the size of the elements of the matrices  $Y$  and  $Z$ .
- (f) Find  $x(0)$  such that  $x(10) = [1 \ 1 \ 1 \ 1]^T$ .