3.4 RECURSIVE DEFINITIONS

EX. DEFINE $f : \mathbb{N} \to \mathbb{Z}$ BY

$$\begin{cases} f(0) = 3 \\ f(n) = 3f(n-1) + 2 & \forall n \geq 1 \end{cases}$$

THEN $f(0) = 3$, $f(1) = 11$, $f(2) = 35$, ...

A RECURSIVE DEFINITION OF $f : \mathbb{N} \to S$ SPECIFIES

- INITIAL CONDITION(S) : $f(0)$
- RECURSIVE FORMULA : A RULE FOR COMPUTING $f(n)$ IN TERMS OF $f(n-1), f(n-2), \ldots, f(0)$.

SOME WELL KNOWN RECURSIVE FUNCTIONS.

EX.

$$\left.\begin{array}{l} g(0) = 1 \\ g(n) = n \cdot g(n-1) \quad \forall n \geq 1 \end{array}\right\} \to g(n) = n!$$

EX. LET $b \in \mathbb{R}$ BE A FIXED CONSTANT.

$$\left.\begin{array}{l} h(0) = 1 \\ h(n) = b \cdot h(n-1) \quad \forall n \geq 1 \end{array}\right\} \to h(n) = b^n.$$

SUCH A FUNCTION (i.e. SEQUENCE) NEED NOT BEGIN AT $n=0$.

Ex. DEFINE $S: \mathbb{Z}^+ \to \mathbb{Z}$ BY

$$\left.\begin{array}{l} S(1) = 1 \\ S(n) = S(n-1) + n \quad \forall n \geq 2 \end{array}\right\} \to S(n) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

THERE MAY BE MORE THAN ONE INITIAL CONDITION. IN GENERAL IF $f(n)$ DEPENDS ON $f(n-k)$ THERE MUST BE AT LEAST k INITIAL TERMS GIVEN.

Ex FIBONACCI SEQUENCE

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2 \end{cases}$$

$\therefore$

$F_0 = 0$ $\quad$ $F_7 = 13$

$F_1 = 1$ $\quad$ $F_8 = 21$

$F_2 = 1$ $\quad$ $F_9 = 34$

$F_3 = 2$ $\quad$ $F_{10} = 55$

$F_4 = 3$ $\quad$ $\vdots$

$F_5 = 5$

$F_6 = 8$

THEOREMS INVOLVING RECURSIVELY DEFINED FUNCTIONS ARE OFTEN PROVED BY INDUCTION.

Ex. SHOW $\forall n \geq 1$

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

i.e. $$\sum_{k=1}^{n} F_{2k-1} = F_{2n}$$

PROOF:

I. $P(1)$ SAYS $F_1 = F_{2\cdot 1}$, i.e. $1 = 1$, WHICH IS TRUE

II. LET $n \geq 1$ AND ASSUME $\sum_{k=1}^{n} F_{2k-1} = F_{2n}$.

THEN

$$\sum_{k=1}^{n+1} F_{2k-1} = \left(\sum_{k=1}^{n} F_{2k-1}\right) + F_{2(n+1)-1}$$

$$= F_{2n} + F_{2n+1} \quad \text{(BY IND. HYP.)}$$

$$= F_{2n+2}$$

$$= F_{2(n+1)}.$$

$\therefore$ RESULT FOLLOWS BY PMI. ///

EX For all $n \geq 1$

$$F_0F_1 + F_1F_2 + F_2F_3 + \cdots + F_{n-1}F_n = \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1}F_{n+1} & (n \text{ odd}) \end{cases}$$

PROOF: Let $P(n)$ be the formula

$$\sum_{k=1}^{n} F_{k-1}F_k = \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1}F_{n+1} & (n \text{ odd}) \end{cases}$$

I. $n=1$ is odd so $P(1)$ says $F_0F_1 = F_0F_2$, i.e. $0 \cdot 1 = 0 \cdot 1$, i.e. $0=0$.

II. Let $n \geq 1$ and assume $P(n)$ holds.

CASE 1: n even $\rightarrow$ $n+1$ odd

ASSUME $P(n)$: $\displaystyle\sum_{k=1}^{n} F_{k-1}F_k = F_n^2$

SHOW $P(n+1)$: $\displaystyle\sum_{k=1}^{n+1} F_{k-1}F_k = F_{(n+1)-1}F_{(n+1)+1}$

$$\sum_{k=1}^{n+1} F_{k-1}F_k = \left(\sum_{k=1}^{n} F_{k-1}F_k\right) + F_nF_{n+1}$$

$$= F_n^2 + F_nF_{n+1} \quad (\text{by ind. hyp.})$$

$$= F_n(F_n + F_{n+1})$$

$$= F_n F_{n+2} \quad \text{(By Fib. Recurrence.)}$$

$\therefore$ $P(n+1)$ is True.

<u>Case 2:</u> n odd $\rightarrow$ $n+1$ even

Assume $P(n)$: $\sum_{k=1}^{n} F_{k-1} F_k = F_{n-1} F_{n+1}$

Show $P(n+1)$: $\sum_{k=1}^{n+1} F_{k-1} F_k = F_{n+1}^2$

$$\sum_{k=1}^{n+1} F_{k-1} F_k = \left(\sum_{k=1}^{n} F_{k-1} F_k\right) + F_{(n+1)-1} F_{n+1}$$

$$= F_{n-1} F_{n+1} + F_n F_{n+1} \quad \text{(By Ind. Hyp.)}$$

$$= (F_{n-1} + F_n) F_{n+1}$$

$$= F_{n+1} \cdot F_{n+1} \quad \text{(By Fib. Recurrence)}$$

$$= F_{n+1}^2 .$$

$\therefore$ $P(n+1)$ is True.

$\therefore$ In either case $P(n) \rightarrow P(n+1)$.

$\therefore$ Result follows from PMI.

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READ LAME'S THEOREM P. 260.

RECURSIVELY DEFINED SETS

TO DEFINE A SET S RECURSIVELY, WE SPECIFY:

(1) AN INITIAL SUBSET S_0 OF S
(2) A RULE (OR RULES) WHICH SHOW HOW TO CONSTRUCT NEW ELEMENTS OF S FROM OLD ONES.

THE SET S THEN CONSISTS OF EXACTLY THOSE ELEMENTS WHICH ARE CONSTRUCTED BY (1) OR (2)

EX. DEFINE $S \subseteq \mathbb{Z}^+$ BY

(1) $5 \in S$
(2) IF $x \in S$ AND $y \in S$, THEN $x + y \in S$

S CERTAINLY CONTAINS $5, 10, 15, 20, 25, \ldots$

WHAT EXACTLY IS THE SET S?

LET $A = \{5n \mid n \in \mathbb{Z}^+\}$, i.e. A IS THE SET OF POSITIVE MULTIPLES OF 5.

WE WILL SHOW THAT $S = A$.

PROOF OF $A \subseteq S$

LET $P(n) =$ '$5n \in S$'.

I. $P(1)$ SAYS $5 \in S$, WHICH IS TRUE BY (1).

II. LET $n \geq 1$ AND ASSUME $P(n)$ IS TRUE, i.e. FOR THIS n, $5n \in S$. THEN $5(n+1) = 5n + 5 \in S$ BY RULE (2).
$\therefore$ $P(n+1)$ IS TRUE.

$\therefore$ $\forall n : 5n \in S$ BY PMI
$\therefore$ $A \subseteq S$.

PROOF OF $S \subseteq A$

OBSERVE THAT A SATISFIES RULES (1) AND (2):

(1) $5 \in A$
(2) $x \in A \wedge y \in A \rightarrow x + y \in A$
(SINCE $5n + 5m = 5(m+n) \in A$.)

THUS ANY NUMBER WHICH CAN BE CONSTRUCTED BY RULES (1) & (2) MUST BELONG TO A. BUT S IS JUST THE SET OF ALL SUCH NUMBERS.

$\therefore S \subseteq A$.

$A \subseteq S$ AND $S \subseteq A$ IMPLIES $S = A$, AS REQUIRED.

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READ: RECURSIVE DEFINITIONS OF STRINGS, WELL FORMED FORMULAS, ROOTED TREES, BINARY TREES, AND STRUCTURAL INDUCTION. P. 262 - 270